Stochastic Heat Equation and Catalytic Super-Brownian Motion

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Von der Fakultät II – Mathematik und Naturwissenschaften
der Technischen Universität Berlin
zur Erlangung des akademischen Grades

Doktor der Naturwissenschaften
– Dr. rer. nat. –

genehmigte Dissertation

Promotionsausschuss:

Vorsitzender:	Prof. Dr. Dirk Ferus
Berichter:	Prof. Dr. Jürgen Gärtner
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Tag der wissenschaftlichen Aussprache: 5. Oktober 2004

Berlin 2004
D 83

Meinem Vater

Ich danke in erster Linie meiner Mutter, die meine mathematische Laufbahn sehr gefördert hat und auf deren Unterstützung ich mich stets verlassen konnte. Ebenso bin ich meiner Schwester Iljana, ihrem Mann Christian Glaßer sowie Herbert Ziezold sehr dankbar, die jederzeit für mich da waren.

Weiterhin danke ich meiner Betreuerin Sylvie Roelly für ihre Unterstützung und Förderung. Ihre mathematischen Kommentare und stilistischen Hinweise haben in besonderem Maße zum Entstehen dieser Arbeit beigetragen.

Für hilfreiche Diskussionen möchte ich mich auch bei Matthias Birkner, Klaus Fleischmann, Axel Simroth und Roger Tribe bedanken. Insbesondere die Gespräche mit Roger Tribe waren für die Erstellung dieser Arbeit sehr anregend.

Während meiner Promotion habe ich die angenehme Atmosphäre im Berliner Graduiertenkolleg "Stochastische Prozesse und Probabilistische Analysis" sehr geschätzt. Ich danke allen Mitgliedern des Kollegs, insbesondere Matthias an der Heiden, Stefan Ankirchner, Steffen Dereich, Pierre-Yves Louis und Christian Vormoor.

In vielerlei Hinsicht habe ich von meinen Gastaufenthalten an den Universitäten von Warwick und Ottawa profitiert. In diesem Zusammenhang bedanke ich mich bei David Elworthy und Roger Tribe bzw. bei Don Dawson, die diese Aufenthalte ermöglichten.

Sehr dankbar bin ich Jürgen Gärtner, dass er die Rolle des Berichters übernommen hat.

Schließlich danke ich der Deutschen Forschungsgemeinschaft sowie dem Marie Curie Programm der Europäischen Union für die finanzielle Unterstützung.

Berlin, August 2004 *Henryk Zähle*

Abstract

This thesis concerns the existence of certain stochastic processes with infinite dimensional state spaces as well as regularity properties of their sample paths.

A main object of interest is the *stochastic heat equation* in $\mathbb{R}^d$ with singular drift and driven by an inhomogeneous time-space white noise. The quadratic variation measure of the white noise is not required to be absolutely continuous w.r.t. the Lebesgue measure, neither in space nor in time. While the homogeneous version of this equation (regular drift, homogeneous white noise) has been studied several times, our general setting has not been investigated so far. We prove the existence of jointly continuous solutions in dimension $d = 1$, provided the drift and the quadratic variation measure of the white noise are not too singular and the coefficients are continuous. In higher dimensions ($d \geq 2$) the disturbance by the white noise is too strong in order to obtain continuous solutions. However, if the noise is skipped, then we can establish that the deterministic heat equation with moderately singular drift possesses jointly continuous solutions in all dimensions $d \geq 1$. For both the stochastic and the deterministic equation statements on uniqueness and non-negativity of solutions will be proven under some additional assumptions on the coefficients.

Another object of interest is the *catalytic super-Brownian motion* in $\mathbb{R}^d$ which arises as (high-density/short-lifetime) measure-valued diffusion limit of a system of d-dimensional branching Brownian particles. The branching time of a particle is governed by the particle's collision with a given time-space measure, the so-called catalyst. We introduce a new admissibility condition for the catalyst and present a direct construction (i.e. without referring to any particle system) of the corresponding catalytic super-Brownian motion $\bar{X} = (\bar{X}_t(dx) : t \geq 0)$. For the sake of completeness we also construct the corresponding branching functional (in the sense of Dynkin) for the approximating particle system. An important feature of the catalytic super-Brownian motion $\bar{X}$ is the characterization as (unique) solution to a certain martingale problem. While it is comparatively easy to verify that $\bar{X}$ solves the martingale problem, it is more delicate to show uniqueness of solutions. In particular, a general uniqueness result was an open problem. By means of a duality argument we prove that uniqueness holds. Further, it will be shown that $\bar{X}$ can be assumed to be continuous w.r.t. the weak topology. Such a regularity statement is already known for more special catalysts. However, the question whether $\bar{X}$ possesses a jointly continuous Lebesgue density field has not been studied so far (except for the case of the classical super-Brownian motion). We show that in dimension $d = 1$ there is a large class of non-atomic catalysts which induce a jointly continuous density field for $\bar{X}$. Moreover, the density field can be characterized as unique solution to the stochastic heat equation (without drift) described above, where the noise coefficient has the shape $a(u) = \sqrt{u}$ and the quadratic variation measure of the time-space white noise coincides with the catalyst.

Zusammenfassung

Diese Arbeit beschäftigt sich mit der Existenz von bestimmten stochastischen Prozessen mit unendlich-dimensionalen Zustandsräumen sowie mit Regularitätseigenschaften von deren Pfaden.

Ein Hauptteil des Interesses gilt der *stochastischen Wärmeleitungsgleichung* in $\mathbb{R}^d$ mit singulärem Drift, die durch ein inhomogenes weißes Zeit-Raum-Rauschen angetrieben wird. Das quadratische Variationsmaß des weißen Rauschens muss dabei weder im Raum noch in der Zeit absolut stetig bezüglich des Lebesguemaßes sein. Während die homogene Variante dieser Gleichung (regulärer Drift, homogenes weißes Rauschen) schon einige Male studiert wurde, ist unser allgemeiner Fall noch nicht untersucht worden. Wir beweisen die Existenz von stetigen Lösungen in Dimension $d = 1$ unter der Voraussetzung, dass der Drift und das quadratische Variationsmaß des weißen Rauschens nicht allzu singulär und die Koeffizienten stetig sind. In höheren Dimensionen ($d \geq 2$) ist die Störung durch das weiße Rauschen zu groß, um stetige Lösungen zu erhalten. Wenn das Rauschen hingegen vernachlässigt wird, dann können wir zeigen, dass die deterministische Wärmeleitungsgleichung mit gemäßigt singulärem Drift stetige Lösungen in allen Dimensionen $d \geq 1$ besitzt. Sowohl für die stochastische als auch für die deterministische Gleichung werden Aussagen über die Eindeutigkeit und die Nichtnegativität von Lösungen unter zusätzlichen Annahmen an die Koeffizienten bewiesen.

Weiterhin beschäftigen wir uns mit der *katalytischen super-Brownschen Bewegung* in $\mathbb{R}^d$, die als maßwertiger Diffusionslimes (hohe Dichte/kurze Lebenszeit) eines Systems von d-dimensionalen verzweigenden Brownschen Teilchen entsteht. Dabei wird der Verzweigungszeitpunkt eines Teilchens durch die Kollision des Teilchens mit einem gegebenen Zeit-Raum-Maß, dem Katalysator, bestimmt. Wir führen eine neue Zulässigkeitsbedingung für den Katalysator ein und präsentieren eine direkte Konstruktion für die zugehörige katalytische super-Brownsche Bewegung $\bar{X} = (\bar{X}_t(dx) : t \geq 0)$. Der Vollständigkeit halber konstruieren wir auch das entsprechende Verzweigungsfunktional (im Sinne von Dynkin) für das approximierende Teilchensystem. Eine wichtige Eigenschaft der katalytischen super-Brownschen Bewegung $\bar{X}$ ist die Charakterisierung als (eindeutige) Lösung eines bestimmten Martingalproblems. Während vergleichsweise einfach verifiziert werden kann, dass $\bar{X}$ das Martingalproblem löst, ist die Eindeutigkeit der Lösung viel schwieriger zu zeigen. Insbesondere war ein allgemeines Eindeutigkeitsresultat ein offenes Problem. Mit Hilfe eines Dualitätsargumentes beweisen wir, dass Eindeutigkeit gilt. Ferner zeigen wir, dass $\bar{X}$ als stetig bezüglich der schwachen Topologie angenommen werden kann. Solch eine Regularitätsaussage ist bereits für speziellere Katalysatoren bekannt. Allerdings wurde die Frage, ob $\bar{X}$ ein stetiges Lebesgue-Dichtefeld besitzt, bisher noch nicht studiert (ausgenommen für den Fall der klassischen super-Brownschen Bewegung). Wir zeigen, dass es in Dimension $d = 1$ eine große Klasse von nicht-atomaren Katalysatoren gibt, die ein stetiges Dichtefeld für $\bar{X}$ induzieren. Ferner kann das Dichtefeld als eindeutige Lösung der oben beschriebenen stochastischen Wärmeleitungsgleichung (ohne Drift) charakterisiert werden, wobei der Koeffizient des Rauschens die Gestalt $a(u) = \sqrt{u}$ hat und das quadratische Variationsmaß des weißen Zeit-Raum-Rauschens mit dem Katalysator übereinstimmt.

Contents

1 Introduction and summary

The heat flow in (rested) media is caused by transport or exchange of energy by chaotically moving molecules. It occurs wherever heat differences exist and acts out until these differences are balanced. From a mathematical point of view the heat flow can be described by means of the *heat equation*

$$\frac{\partial}{\partial t}u(t,x) = \mathsf{c}\Delta u(t,x) + \mathsf{p} \qquad (t \geq 0,\ x \in \mathbb{R}^d) \tag{1.1}$$

where Δ denotes the Laplacian. Here $u(t,x)$ displays the temperature at site x at time t, c denotes the heat conductivity of the medium and p represents the heat performance of the medium. If the parameters c and p depend also on time and/or space, then equation (1.1) corresponds to the heat flow in an inhomogeneous medium. In this thesis we always assume c to be constant and allow $\mathsf{p} = \mathsf{p}(t,x) = \mathsf{p}(t,x,u(t,x))$ not only to vary, possibly randomly, in time and space but also to depend on the temperature. In the above setting, the parameter c should be positive. On the other hand, $\mathsf{p}(t,x)$ may be positive or negative depending on whether site x acts as a heat source or a heat sink, respectively, at time t.

If $\mathsf{c} = 1/2$ and $\mathsf{p} \equiv 0$, then the (deterministic) heat flow corresponding to (1.1) can be approximated by a system of independent Brownian particles in $\mathbb{R}^d$ through a high density limit. The Brownian particles can be interpreted as the chaotically moving molecules which are responsible for the transport of energy, hence for the heat flow. In this case the Cauchy problem (1.1) with initial condition $u(0,.) = \eta(.) \in C_b^2(\mathbb{R}^d)$ has a unique solution which can be written down explicitly. In fact, the solution is given by $u(t,x) := P_t\eta(x)$ where (P_t) denotes the heat semigroup which is determined by the Gaussian kernel.

The situation may change drastically when considering some non-trivial p. In this thesis we focus on existence and uniqueness of jointly continuous solutions to equation (1.1) with $\mathsf{c} = 1/2$ and rather general p. In fact, we consider the *stochastic heat equation*

$$\frac{\partial}{\partial t}u(t,x) = \frac{1}{2}\Delta u(t,x) + \underbrace{b(t,x,u(t,x))\frac{\sigma(dtdx)}{dtdx}(t,x) + a(t,x,u(t,x))\dot{w}^{\varrho}(t,x)}_{\mathsf{p}} \tag{1.2}$$

where $\frac{\sigma(dtdx)}{dtdx}$ denotes the Lebesgue density of a positive Radon measure $\sigma(dtdx)$ and $\dot{w}^{\varrho}$ is a time-space white noise with intensity measure $\varrho(dtdx)$, i.e., formally, the mixed partial derivative $\frac{\partial^{1+d}}{\partial t\partial x_1 \cdots \partial x_d}w^{\varrho}$ of an inhomogeneous Brownian sheet $(w^{\varrho}(t,x) : t \geq 0, x \in \mathbb{R}^d)$ based on a positive Radon measure $\varrho(dtdx)$; for a precise definition see Section 5.3.

In engineering and natural science (e.g. in acoustics) the term *noise* is associated with a signal having a wide range of frequencies and random amplitudes. A noise is said to be *white* if it is composite of harmonic vibrations (of "all" frequencies) whose random amplitudes are independent and have the same intensity. To some extent, white noises are subject to the "maximal disorder". From a mathematical point of view a *(time) white noise* is regarded as a stochastic process $\dot{w} = (\dot{w}_t : t \geq 0)$ satisfying: (a) $\dot{w}$ is a stationary process; (b) $\mathbb{E}[\dot{w}_t] = 0$ for all t; (c) $\dot{w}_t, \dot{w}_{t'}$ independent for $t \neq t'$. Heuristic arguments suggest to associate (formally) $\dot{w}_t$ with the derivative $\frac{d}{dt}w_t$ of a Brownian motion w at t. Indeed, let us look for a process $w = (w_t : t \geq 0)$ which satisfies $w_{t+h} - w_t = \dot{w}_t h$ for $h > 0$. Properties (a)–(c) imply that w must have stationary independent increments with mean zero; and Brownian motion is the only continuous process with that properties. However, $\lim_{h\downarrow 0}(w_{t+h} - w_t)/h$ is known to be degenerated. More generally, with a *time-space white noise* $\dot{w}^\varrho = (\dot{w}^\varrho(t,x) : t \geq 0, x \in \mathbb{R}^d)$ with intensity measure $\varrho(dtdx)$ we associate the mixed partial derivative of an inhomogeneous Brownian sheet $w^\varrho = (w^\varrho(t,x) : t \geq 0, x \in \mathbb{R}^d)$, i.e. $\dot{w}^\varrho(t,x) = \frac{\partial^{1+d}}{\partial t \partial x_1 \cdots \partial x_d} w^\varrho(t,x)$. For $d = 1$, for instance, w^ϱ is given by

$$w^\varrho(t',x') - w^\varrho(t',x) - w^\varrho(t,x') + w^\varrho(t,x) = \bar{W}^\varrho((t,t'] \times (x,x'])$$

where $\bar{W}^\varrho$ is a so-called *white noise measure* based on ϱ, i.e. a random set function with: $\bar{W}^\varrho(A) \sim N(0,\varrho(A))$; $\bar{W}^\varrho(A), \bar{W}^\varrho(A')$ independent and $\bar{W}^\varrho(A \cup A') = \bar{W}^\varrho(A) + \bar{W}^\varrho(A')$ for any disjoint sets $A, A' \in \mathcal{A}([0,\infty) \times \mathbb{R}^d)$. Of course, $\dot{w}^\varrho$ does not exist. But if it would exist, then we had $\dot{w}^\varrho(t,x)dtdx = w^\varrho(t+dt, x+dx) - w^\varrho(t+dt,x) - w^\varrho(t,x+dx) + w^\varrho(t,x) = \bar{W}^\varrho((t,t+dt] \times (x,x+dx])$. So one might think of $\dot{w}^\varrho$ as the density of the "measure" $\bar{W}^\varrho$; especially $\dot{w}^\varrho \equiv 0$ outside the closed support of ϱ.

The formulation of the stochastic partial differential equation (SPDE) in (1.2) is rather vague. On the one hand, neither $\varrho(dtdx)$ nor $\sigma(dtdx)$ will be required to be absolutely continuous w.r.t. the Lebesgue measure $dtdx$; in particular the Lebesgue density of $\sigma(dtdx)$ might fail to exist. On the other hand, w^ϱ is not differentiable, at least not everywhere. The way out is to regard SPDE (1.2) with initial condition $u(0,.) = \eta(.)$ as:

$$\begin{aligned} \langle u(t,.),\psi\rangle \;&=\; \langle \eta,\psi\rangle + \int_0^t \langle u(r,.), \frac{1}{2}\Delta\psi\rangle dr \qquad\qquad (t \geq 0,\ \psi \in C_c^\infty(\mathbb{R}^d)) \qquad (1.3) \\ &+ \int_0^t \int_{\mathbb{R}^d} b(r,y,u(r,y))\psi(y)\sigma(drdy) + \int_0^t \int_{\mathbb{R}^d} a(r,y,u(r,y))\psi(y) W^\varrho(drdy) \end{aligned}$$

where $\langle \phi, \psi \rangle$ denotes the integral $\int \phi(x)\psi(x)dx$. The last term on the r.h.s. of (1.3) is a *Walsh integral* against an orthogonal martingale measure $W^\varrho(dtdx)$ with quadratic variation measure $\varrho(dtdx)$. In **Chapter 5**, especially in Section 5.3, we thoroughly review Walsh's stochastic integration theory ([Wal86]). In particular we give a detailed justification for the relationship between (1.2) and (1.3). If $\varrho(dtdx) = \sigma(dtdx) = dtdx$, then $\frac{\sigma(dtdx)}{dtdx} \equiv 1$ and $\dot{w}^\varrho$ gets a standard time-space white noise $\dot{w}$. In that case SPDE (1.2) has been studied several times for space dimension $d = 1$ w.r.t. existence and uniqueness of jointly continuous solutions ([Iwa87], [MP92], [Shi94], [Myt98] and others; the restriction to dimension $d = 1$ is due to the disturbing impact of $\dot{w}$ which excludes continuous solutions in higher dimensions). On the other hand, the case where ϱ is a singular measure has not been studied so far. So we asked the questions: Under which assumptions on ϱ and σ can equation (1.2) – in the sense of (1.3) – have a strong/weak jointly continuous solution? When is the solution strongly/weakly unique? When is it non-negative? The motivation comes from the theory of catalytic super-Brownian motion which will be discussed below. Intuitively, a continuous solution should exist if ϱ and σ are not too singular.

A benchmark for the singularity of a Borel measure $\mu(dy)$ on $\mathbb{R}^n$ is the behavior of $\mu(B[y,r])$ as $r \downarrow 0$ where $B[y,r]$ denotes the closed ball around y with radius $r > 0$. Suppose $\mu(B[y,r]) \sim r^\gamma$ for small r and some $\gamma \in [0,n]$. Then a small γ indicates a strong concentration of μ-mass around y. To exclude a strong mass concentration around y one can require the existence of some "large" γ so that $\mu(B[y,r]) \leq c\, r^\gamma$ $\forall r \in (0,1]$. This implies in particular that the Hausdorff dimension of μ's support is at least γ.

In general we need to restrict to space dimension $d = 1$ because of the strong disturbing impact of the white noise.[1] If we skip the noise, then continuous solutions can be obtained even in higher dimensions. Hence, $\varrho(dtdx)$ will always be assumed to be a Radon measure on $[0,\infty) \times \mathbb{R}$ whereas $\sigma(dtdx)$ may be defined on $[0,\infty) \times \mathbb{R}^d$. We further assume $\varrho(dtdx)$ and $\sigma(dtdx)$ to possess decompositions $\varrho_1(t,dx)\varrho_2(dt)$, respectively $\sigma_1(t,dx)\sigma_2(dt)$, for which there exist $\alpha_1, \alpha_2 \in [0,1]$ and $\beta_1 \in [0,d], \beta_2 \in [0,1]$ such that $\alpha_1/2 + \alpha_2 > 1$, $\beta_1/2 + \beta_2 > d/2$ and for every $T > 0$:

(A) $\quad \sup_{t\leq T} \sup_{x\in\mathbb{R}} \varrho_1(t, B[x,r]) \leq c_T\, r^{\alpha_1} \quad$ and $\quad \sup_{t\leq T} \varrho_2(B[t,r]) \leq c_T\, r^{\alpha_2}$

(B) $\quad \sup_{t\leq T} \sup_{x\in\mathbb{R}^d} \sigma_1(t, B[x,r]) \leq c_T\, r^{\beta_1} \quad$ and $\quad \sup_{t\leq T} \sigma_2(B[t,r]) \leq c_T\, r^{\beta_2}$

for all $r \in (0,1]$. Examples for measures satisfying (A) and (B) are presented in Section 2.8. For instance, one can associate "Cantor-type" measures. In **Chapter 6** we focus on $d = 1$ and establish continuous solutions to SPDE (1.2) under conditions (A) and (B). For Lipschitz continuous coefficients we can find strongly unique strong solutions. The key is a Picard-Lindelöf iteration for which we need to generalize Gronwall's lemma. In the non-Lipschitz case we obtain at least weak solutions; the crux is a tightness argument. The question of uniqueness of solutions for non-Lipschitz coefficients is quite delicate. We prove weak uniqueness only for $a(t,x,u) = \sqrt{|u|}$ and $b \equiv 0$ by means of a duality argument. This is admittedly only a single case but quite interesting in the context of catalytic super-Brownian motion. If SPDE (1.2) is wanted to describe the evolution of a population system, then it is a natural desire to obtain non-negativity of solutions. Under some additional assumptions on the coefficients we are able to show non-negativity. In **Chapter 7** we allow the space dimension d to be arbitrary but we skip the noise ($a \equiv 0$). Hence equation (1.2) turns into a deterministic PDE. Under condition (B) we establish the analogous results on existence, uniqueness and non-negativity of continuous solutions. We also focus on the backward equation whose solutions induce an inhomogeneous semigroup of operators on $C(\mathbb{R}^d)$. Equation (1.2) with $a \equiv 0$ has been considered earlier (see e.g. [FG86], [DF92]) but under more restrictive assumptions on σ and b. In particular $\sigma_2(dt)$ was assumed to be the Lebesgue measure dt. Note that, if $\sigma_2(dt) = dt$, each measure $\sigma_1(t,dx)$, $t \geq 0$, must be supported by a set with Hausdorff dimension strictly greater than $d-2$; otherwise σ cannot satisfy (B). In particular, $\sigma_1(t,dx)$ is ruled out to be a Dirac measure for $d \geq 2$.

Another object of interest in this thesis is the so-called *catalytic super-Brownian motion* which is a generalization of the classical super-Brownian motion (SBM). Its evolution strongly depends on its *catalyst* $\varrho(dtdx)$, which is a positive measure on $[0,\infty) \times \mathbb{R}^d$. More precisely, the catalytic SBM describes the evolution of the finite measure-valued high-density/short-lifetime limit ($n \to \infty$) of the following branching particle system:

[1] If the solution is not required to be continuous but only to be function-valued, then the situation may look completely different. Therefore we stress the fact that we shall study only continuous solutions.

- Each new born particle is given a lifetime ζ by an exponential distribution with parameter n. The lifetimes are independent.

- The particles move independently according to Brownian motion through $\mathbb{R}^d$ and carry mass $1/n$ each.

- Suppose a particle is born at time s. Then the particle dies when its individual "branching age" $A = (A(s,t] : t \geq s)$ first exceeds the lifetime ζ. At that time the particle performs a critical binary branching event. The offspring are independent copies of the parent and are initially located at the parent's death site.

Here the individual "branching age" A, i.e. the (additive) branching functional, is given by the collision process of the (Brownian) particle B and the catalyst ϱ. Informally, the collision process A increases continuously when B is moving on the support of ϱ and does not vary otherwise. In other words, the more intense the collision of B and ϱ the faster does A increase. In **Chapter 8** we discuss the notion of collision processes in detail. In [EP94] and [Del96] collision processes were constructed as continuous additive functionals (CAF) of Brownian motion in the sense of $A_{s+t} = A_s + A_t \circ \theta_s$, where θ denotes the usual shift operator on the paths space. To do so it was necessary to assume ϱ to be Lebesgue in time. However, we intend to construct A for catalysts ϱ that might be singular in time. Therefore we have to adopt Dynkin's more general notion of CAF ([Dyn94]) in order to define and to construct the collision process. It is natural to ask: For which measures ϱ does the collision process exist? Intuitively, ϱ should not be too singular. Otherwise the Brownian motion would not "meet" ϱ's mass. Condition (B) turns out to be a proper indicator. In fact, we shall construct the collision process A for each measure ϱ that satisfies (B) (with σ replaced by ϱ). We stress the fact that condition (B) provides a new admissibility condition for catalysts of the catalytic SBM (in particular it generalizes Delmas' hypothesis (H) from [Del96]). In fact, under condition (B) the collision process can be shown to be a *branching functional* in the sense of [Dyn94]. This guarantees that the limit (=catalytic SBM) of the above particle system can be taken; cf. [Daw93] or [Dyn94]. One also refers to the limit as $(B, A, (.)^2)$-superprocess.

The catalytic SBM $\bar{X} = (\bar{X}_t(dx) : t \geq 0)$ with catalyst ϱ satisfying condition (B) (with σ replaced by ϱ), which will be thoroughly studied in **Chapter 9**, is related to the heat equation (1.2) in two different ways. On the one hand, $\bar{X}$ is a Markov process and so it can be characterized via the Laplace transforms of its transition probabilities $\mathbb{P}_{s,\eta}[\bar{X}_t \in .]$. For any finite measure $\eta(dx)$ and $t \geq s$ the Laplace transform has the shape

$$\mathbb{E}_{s,\eta}\left[e^{-\langle \bar{X}_t, \psi\rangle}\right] = e^{-\langle \eta, U_{s,t}\eta(.)\rangle}, \qquad \psi \in C_b^+(\mathbb{R}^d), \tag{1.4}$$

where $(U_{s,t}\psi(x) : s \in [0,t], x \in \mathbb{R}^d)$ is the unique solution to the (deterministic) backward version of equation (1.2) with $a \equiv 0$, $b(t,x,u) = -\frac{1}{2}u^2$, final condition $u(t,.) = \psi(.)$ and σ replaced by ϱ, cf. (9.7). We carry out a direct construction of $\bar{X}$ via (1.4) where we use results from Chapter 7 and [Fit88] as well as general results on positive and negative definite operators. On the other hand, equation (1.2) with $a(t,x,u) = \sqrt{u}$ and $b \equiv 0$, i.e.

$$\frac{\partial}{\partial t} X_t(x) = \frac{1}{2}\Delta X_t(x) + \sqrt{X_t(x)}\,\dot{w}^{\varrho}(t,x), \tag{1.5}$$

describes the evolution of $\bar{X}$, formally at least. The Laplacian component corresponds to the diffusion of the infinitesimal Brownian particles. The noise term is to be associated with the branching of the particles, and its shape can be motivated by Feller's branching diffusion which arises as the diffusion limit of discrete Galton-Watson branching processes. When can equation (1.5) – in the sense of (1.3) – rigorously be related to the catalytic SBM $\bar{X}$? One can show that this is the case if $\bar{X}$ has a jointly continuous Lebesgue density field X. The key is the characterization of $\bar{X}$ as unique solution to the martingale problem:

$$M_t(f) = \langle \bar{X}_t, f(t,.)\rangle - \langle \eta, f(0,.)\rangle - \int_0^t \langle \bar{X}_r, \tfrac{1}{2}\Delta f(r,.) + \tfrac{\partial}{\partial r} f(r,.)\rangle dr$$

$$\langle M(f)\rangle_t = \int_0^t \int_{\mathbb{R}^d} f^2(r,y) C_{[\bar{X},\varrho]}(drdy)$$

where $M_.(f)$ has to be a continuous square-integrable martingale with quadratic variation process $\langle M(f)\rangle$ for every $f \in C^{1,2}_{b,\infty}([0,\infty)\times\mathbb{R}^d)$; $\eta(dx)$ is the initial value, $C_{[\bar{X},\varrho]}$ is the collision measure of $\bar{X}_t(dx)dt$ and $\varrho(dtdx)$ and $\langle \mu, \psi\rangle$ denotes the integral $\int \psi(x)\mu(dx)$. If $\varrho(dtdx) = dtdx$ (classical case), then $C_{[\bar{X},\varrho]}(dtdx) = \bar{X}_t(dx)dt$ and the martingale problem was solved w.r.t. existence and uniqueness of solutions in [RC86]. For general (time-constant) catalysts Delmas ([Del96]) showed that the catalytic SBM solves the martingale problem but the question of uniqueness remained open. We do not only generalize Delmas' result but we also prove uniqueness of solutions by means of a duality argument.

Now it is natural to ask for which catalysts ϱ does $\bar{X}$ have a jointly continuous Lebesgue density field. In dimension $d = 1$ the classical SBM ($\varrho(dtdx) = dtdx$) possesses a continuous density field ([KS88], [Rei89]). On the other hand, in the case of a single point catalyst $\varrho(dtdx) \equiv \delta_c(dx)dt$, $c \in \mathbb{R}$ fixed, the 1-dimensional catalytic SBM $\bar{X}$ does not have a jointly continuous density field on the whole product space $[0,\infty)\times\mathbb{R}$. It is true that there is a density field X which is jointly continuous on $(0,\infty)\times\mathbb{R}\setminus\{c\}$, cf. Theorem 1.2.2 of [DF94]. But at random exceptional times t the density $X_t(.)$ blows up when approaching the catalyst's position c. Indeed, as shown in [FLG95] (p.82), a.s. there is a dense set $\mathcal{T} \subset [0,\infty)$ and a constant $K > 0$ such that

$$\limsup_{x\to c} \frac{X_t(x)}{\log\log(1/|x-c|)} \geq K \qquad \forall t \in \mathcal{T}. \tag{1.6}$$

Can one get back to the joint continuity of X on the whole product space $(0,\infty)\times\mathbb{R}$ by "smearing out" the mass of $\delta_c(dx)$ around c? One might guess that a slight smoothing of atom mass leads only to sharp peaks of the density X rather than to blow ups (in sense of (1.6)). This is indeed the case. In Section 9.8 we show the existence of a jointly continuous density field X whenever the catalyst ϱ satisfies condition (A) (note that condition (A) is stronger than (B)), and in Section 9.9 we characterize X as unique solution to SPDE (1.5). It should be mentioned that atomic measures violate (A) but nearly all non-atomic measures satisfy it. In particular, for any $0 < \alpha \leq 1$ one can find a catalyst which satisfies (A) and is supported by a set with Hausdorff dimension α. In higher dimension $d \geq 2$ the situation is not completely clear. On the one hand, the states of the classical SBM in $\mathbb{R}^d$, $d \geq 2$, are known to be singular w.r.t. the Lebesgue measure, for fixed times at least ([DH79]). On the other hand, the states may get "smoother" when introducing some singular catalyst as the following result for $\varrho(dtdx) \equiv \varrho_1(dx)dt$ and $d \geq 1$ shows (cf. [Del96], Théorème 8.1): If the closed support $\mathrm{supp}(\varrho_1)$ of $\varrho_1(dx)$ has Lebesgue measure zero, then the states $\bar{X}_t(dx)$ possess Lebesgue densities $X_t(.)$. Moreover, on the complement $\mathrm{supp}(\varrho_1)^c$ of $\mathrm{supp}(\varrho_1)$ these densities are smooth and solve the heat equation

$$\frac{\partial}{\partial t} X_t(x) = \frac{1}{2}\Delta X_t(x), \qquad (t,x) \in (0,\infty) \times \mathrm{supp}(\varrho_1)^c.$$

Nevertheless one might guess that despite that strong regularity off the catalyst the density X occasionally blows up (similar to (1.6)) on the boundary of $\mathrm{supp}(\varrho_1)^c$ if $d \geq 2$. Anyway, in dimension $d = 1$ our result shows that under condition (A) the density X extends continuously to all of $(0,\infty) \times \mathbb{R}$.

The proofs of the mentioned results of Chapter 9 rely on properties of the catalytic SBM $\bar{X}$ (moment formulae, sample continuity, existence of collision measure of $\bar{X}$ and the catalyst) which are already known for more special catalysts (cf. [Del96], [DF97]). In Sections 9.4 – 9.6 we thoroughly prove these properties for our general catalysts. We will also focus on the strong Markov property of the catalytic SBM $\bar{X}$.

Chapters 2 is devoted to measure theoretic foundations including the basics of the theory of Hausdorff measures and dimension. We also define conditions (A) and (B) exactly and present some examples. In **Chapter 3** we review important results on stochastic processes which will be needed throughout this thesis. In Sections 3.3 and 3.5 we prove criteria for $C_{tem}(\mathbb{R}^d)$-valued continuity, respectively tightness in $C([0,\infty), C_{tem}(\mathbb{R}^d))$, which were stated (without proofs) in [Shi94] for $d = 1$. In particular, we establish criteria for relative compactness of subsets of $C([0,\infty), C_{tem}(\mathbb{R}^d))$. In **Chapter 4** we give a number of auxiliary lemmas concerning measure potentials, the heat kernel, the heat semigroup and generalized Gronwall lemmas.

2 Measure theoretic foundations

In this chapter we recall some basics of measure theory. A particular goal is to specify the closed support of Borel measures on $\mathbb{R}^n$. While, for instance, the Lebesgue measure and the Dirac measure are supported by easy geometric sets, there are plenty of measures whose closed supports have a more complicated (fractal) structure. A basic notion of differentiating the "fractal character" of sets is the *Hausdorff dimension*, initiated by Hausdorff in 1919. We will give a brief introduction into the theory of Hausdorff measures and dimension and we recall a basic method for determining a lower bound for the Hausdorff dimension of a set (mass distribution principle). We shall also study a fundamental example. Further points of interest are the identification and the convergence of measures. In particular we introduce the vague and the weak topology. Finally we will focus on the notion of kernels and we define two classes of Borel measures on $[0,\infty)\times\mathbb{R}$, respectively $[0,\infty)\times\mathbb{R}^d$, which play a central role in this thesis (conditions (A) and (B)).

2.1 Definitions and basics

Let S be an arbitrary set and denote the system of all subsets of S by $\mathfrak{P}(S)$. In particular, $\mathfrak{P}(S)$ is a σ-algebra. If $\mathcal{G}$ is a subsystem of $\mathfrak{P}(S)$, then we write $\sigma(\mathcal{G})$ for the coarsest σ-algebra which contains all sets from $\mathcal{G}$. It is called the σ-algebra *generated by* $\mathcal{G}$. In the case of a topological space S, the σ-algebra generated by the system of all open sets in S is denoted by $\mathcal{B}(S)$ and called *Borel σ-algebra* in S. Now suppose I is some arbitrary index set and consider, for every $i\in I$, a measurable space $[S_i,\mathcal{S}_i]$ and a map $f_i : S\to S_i$. Then the σ-algebra $\sigma(f_i^{-1}(\mathcal{S}_i) : i\in I)$ is said to be *generated by* the maps f_i, $i\in I$. A map $\mu:\mathfrak{P}(S)\to[0,\infty]$ is called *outer measure* if

(*i*) $\mu(\emptyset)=0$,

(*ii*) $\mu(A)\le\sum_{i=1}^{\infty}\mu(A_i)\quad\forall A,A_i\in\mathfrak{P}(S)$ with $A\subset\cup_{i=1}^{\infty}A_i$ ("sub-σ-additivity").

If such a map μ is wanted to be understood as a measure for the content of sets, then it is a natural desire to obtain "=" in (*ii*) for every $A\in\mathfrak{P}(S)$ and every disjoint system $(A_i)_{i=1}^{\infty}\subset\mathfrak{P}(S)$ with $A=\cup_{i=1}^{\infty}A_i$. However, this will not be possible in general when considering sets from $\mathfrak{P}(S)$. The way out is to restrict the map μ to the system $\mathfrak{P}_\mu(S)$ of all μ-measurable sets. Here a set $B\in\mathfrak{P}(S)$ is said to be *μ-measurable* if

$$\mu(B)=\mu(A\cap B)+\mu(B\setminus A)\qquad\forall A\in\mathfrak{P}(S).$$

$\mathfrak{P}_\mu(S)$ can be shown to form a σ-algebra in S. Also, μ on $[S,\mathfrak{P}_\mu(S)]$ is a measure. Recall that a map $\mu:\mathcal{S}\to[0,\infty]$ on a σ-algebra $\mathcal{S}$ on S is said to be a *measure* on $[S,\mathcal{S}]$ if

(*i*) $\mu(\emptyset)=0$,

(*ii*) $\mu(\cup_{i=1}^{\infty}A_i)=\sum_{i=1}^{\infty}\mu(A_i)\quad\forall$ disjoint systems $(A_i)_{i=1}^{\infty}\subset\mathcal{S}$ ("σ-additivity").

A map $\mu:\mathcal{S}\to(-\infty,\infty)$ satisfying conditions (*i*) and (*ii*) is called a *signed measure.* Note that a finitely additive set function $\mu:\mathcal{S}\to[0,\infty)$ is σ-additive (i.e. has property (*ii*)) if and only if it is *continuous from above* (i.e. $\mu(A) = \lim_{i\to\infty} \mu(A_i)$ holds for all $A \in \mathcal{S}$ and $(A_i)_{i=1}^{\infty} \subset \mathcal{S}$ with $A_{i+1} \subset A_i$, $A = \cap_{i=1}^{\infty} A_i$ and $\mu(A_i) < \infty$), cf. Satz 3.2 of [Bau92]. Let μ and ν be two measures on $[S, \mathcal{S}]$. Then μ is said to be *absolutely continuous* w.r.t ν if, for every $A \in \mathcal{S}$, $\nu(A) = 0$ implies $\mu(A) = 0$. Otherwise μ is said to be *singular* w.r.t. ν. A measure space $[S, \mathcal{S}, \mu]$ is called *complete* if $\mathcal{S}$ contains all μ-negligible sets; a *μ-negligible set* is defined to be a subset of S which is contained in a *μ-null set*, i.e. in a set $A \in \mathcal{S}$ with $\mu(A) = 0$. The classes of μ-null sets and μ-negligible sets are denoted by N_μ, respectively $\mathcal{N}_\mu$. If $[S, \mathcal{S}, \mu]$ is not complete, it can be completed by replacing $\mathcal{S}$ by $\tilde{\mathcal{S}}^\mu := \sigma(\mathcal{S} \cup \mathcal{N}_\mu)$ and μ by its extension $\tilde{\mu}$ from $\mathcal{S}$ to $\tilde{\mathcal{S}}^\mu$. $\tilde{\mathcal{S}}^\mu$ is called the *μ-completion* of $\mathcal{S}$. If S is a topological space, then an outer measure is said to be *Borel* if $\mathcal{B}(S) \subset \mathfrak{P}_\mu(S)$. In particular, a Borel outer measure on $[S, \mathcal{B}(S)]$ is a measure and called *Borel measure* on S. The restriction to the Borel σ-algebra $\mathcal{B}(S)$ simplifies the study of (outer) measures.

Although we could work with a general metric space S, we assume $S = \mathbb{R}^n$ throughout Sections 2.2 – 2.4. For our purposes this is completely sufficient.

2.2 Carathéodory's construction

In this section we will see how to extend a premeasure to a Borel outer measure. Let $\mathcal{C}$ be a system of subsets of $\mathbb{R}^n$ such that $\emptyset \in \mathcal{C}$, and $\phi : \mathcal{C} \to [0, \infty]$ be a premeasure, i.e. $\phi(\emptyset) = 0$. The *diameter* of a non-empty set $A \subset \mathbb{R}^n$ is defined as $|A| := \sup\{|a - a'| : a, a' \in A\}$. A system $(C_i) \equiv (C_i)_{i\in I} \subset \mathcal{C}$ is called $(\mathcal{C}, \delta)$-*covering* of a set $A \subset \mathbb{R}^n$ if I is a countable index set, $A \subset \cup_{i\in I} C_i$ and $|C_i| \le \delta$ for all $i \in I$. Assume $\mathcal{C}$ is chosen in such a manner that for every $\delta > 0$ there is at least one $(\mathcal{C}, \delta)$-*covering* of $\mathbb{R}^n$. Set

$$\mu_\delta(A) := \inf\Big\{ \sum_{i\in I} \phi(C_i) : (C_i) \text{ is a } (\mathcal{C}, \delta)\text{-covering of } A \Big\}, \quad A \subset \mathbb{R}^n,$$

$$\mu(A) := \lim_{\delta\downarrow 0} \mu_\delta(A), \quad A \subset \mathbb{R}^n. \tag{2.1}$$

The limit exists in $[0, \infty]$ since $\mu_\delta(A)$ is clearly non-decreasing (as $\delta \downarrow 0$) for every A. The set function μ_δ is easily seen to be an outer measure, and it can be deduced that the same is true for μ. While μ_δ usually fails to be a *Borel* outer measure, μ is always Borel:

Theorem 2.1 [Carathéodory's construction] *μ defined in (2.1) is a Borel outer measure. In particular, μ on $[\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)]$ is a Borel measure.*

(For a proof see page 55 of [Mat95].) Carathéodory's construction method can be used to define the (outer) Lebesgue measure $dx = \mathcal{L}^n$ on $\mathbb{R}^n$. Let $\mathcal{R}$ be the system of all rectangle $R = (a_1, b_1) \times \ldots \times (a_n, b_n)$ in $\mathbb{R}^n$ and set $V^n(R) := \prod_{i=1}^n (b_i - a_i)$. Further, $V^n(\emptyset) := 0$.

Example 2.2 [Lebesgue measure] *The Borel outer measure $\mathcal{L}^n$ arosen out of the Carathéodory construction with $\mathcal{C} := \mathcal{R} \cup \{\emptyset\}$ and $\phi(R) := V^n(R)$, $R \in \mathcal{R} \cup \{\emptyset\}$, is called outer Lebesgue measure. In particular, $\mathcal{L}^n$ on $[\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)]$ is a Borel measure and called Lebesgue measure. It can be shown that $\mathcal{L}^n(R) = V^n(R)$ holds for all $R \in \mathcal{R}$.*

2.3 Hausdorff measures and dimension

In this section we use Carathéodory's construction to define the *α-dimensional Hausdorff measure.* We also introduce the notion of Hausdorff dimension. Set $0^0 := 1$ and $|\emptyset|^\alpha := 0$.

Definition 2.3 [HAUSDORFF MEASURE] *For $\alpha \geq 0$, the Borel outer measure $\mathcal{H}^\alpha$ arosen out of the Carathéodory construction with $\mathcal{C} := \mathfrak{P}(\mathbb{R}^n)$ and $\phi(C) := |C|^\alpha$, $C \in \mathfrak{P}(\mathbb{R}^n)$, is called outer α-dimensional Hausdorff measure. In particular, $\mathcal{H}^\alpha$ on $[\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)]$ is a Borel measure and called α-dimensional Hausdorff measure.*

It is easy to see that $\mathcal{H}^0$ is just the counting measure, i.e. $\mathcal{H}^0(A) = \#A$. The next result provides further basic properties of Hausdorff measures (cf. [Mat95], p.56-58).

Proposition 2.4 [PROPERTIES] *For all $\alpha \geq 0$, $a \in \mathbb{R}^n$, $0 < r < \infty$ and $A \subset \mathbb{R}^n$ we have:*

(i) $\mathcal{H}^n(A) = (2^n/v_n)\, \mathcal{L}^n(A)$

(ii) $\mathcal{H}^\alpha(A) < \infty \Rightarrow \mathcal{H}^{\alpha'}(A) = 0 \quad \forall \alpha' > \alpha$ *("jump property")*

(iii) $\mathcal{H}^\alpha(A + a) = \mathcal{H}^\alpha(A)$ *("translation invariance")*

(iv) $\mathcal{H}^\alpha(rA) = r^\alpha \mathcal{H}^\alpha(A)$ *("dilation invariance")*

where $A + a = \{x + a : x \in A\}$, $rA = \{rx : x \in A\}$ and v_n =volume of the unit ball in $\mathbb{R}^n$.

Note that $\mathcal{H}^\alpha \equiv 0$ for $\alpha > n$ by (i) and (ii). Part (ii) also justifies the following definition.

Definition 2.5 [HAUSDORFF DIMENSION] *The Hausdorff dimension of a set $A \subset \mathbb{R}^n$ is defined by* $\dim A := \sup\{\alpha \geq 0 : \mathcal{H}^\alpha(A) = \infty\} = \inf\{\alpha \geq 0 : \mathcal{H}^\alpha(A) = 0\}$.

Plainly, $\dim A \in [0, n]$ for all $A \subset \mathbb{R}^n$. Be aware that the definition of $\dim A$ does not give any information about the value of $\mathcal{H}^{\dim A}(A)$. In fact this expression can take any value in $[0, \infty]$. Two crucial properties are given in the next proposition (cf. [Mat95], p.59).

Proposition 2.6 [PROPERTIES] *For all $A, A', A_1, A_2, \ldots \subset \mathbb{R}^n$ we have:*

(i) $A \subset A' \Rightarrow \dim A \leq \dim A'$ *("monotonicity")*

(ii) $\dim \cup_{i=1}^\infty A_i = \sup_{i \geq 1} \dim A_i$ *("σ-stability").*

The Hausdorff dimension provides a benchmark for the "thickness" of a set. Let us justify this, at least formally, for a *bounded* set $A \subset \mathbb{R}^n$. If the Hausdorff dimension of A is large, then the jump of $(\mathcal{H}^\alpha(A) : \alpha \geq 0)$ occurs "latish". That means we can find "large" α such that, for each $(\mathfrak{P}(\mathbb{R}^n), \delta)$-covering (C_i) of A, the sum $\sum_{i \in I} |C_i|^\alpha$ is "close to ∞" when $\delta > 0$ is "close to 0". However, if α is "large" and δ is "close to 0", then $|C_i|^\alpha$ is "small". And so $\sum_{i \in I} |C_i|^\alpha$ being "close to ∞" implies that we need "many" sets with diameter $\leq \delta$ to cover A. This indicates that A must be "thick" in some sense.

In many cases it is relatively easy to find an upper bound for the Hausdorff dimension since one "only" has to find an efficient covering of the set. On the other hand, it looks more difficult to find a lower bound. A basic principle for determining a lower bound is described in the next section.

2.4 Mass distribution principle

An intuitive motivation for the *mass distribution principle* is the following. If it is possible to distribute a positive amount of mass on a set A in such a manner that its local concentration is bounded from above, then the set A must be large in some sense. Note that a Borel measure μ on $\mathbb{R}^n$ (or its restriction $\mu(.\cap A)$) is called a *mass distribution* on a Borel set $A \in \mathcal{B}(\mathbb{R}^n)$ if $0 < \mu(A) < \infty$. Let $B[x,r]$ denote the closed ball in $\mathbb{R}^n$ around x with radius r.

Proposition 2.7 [MASS DISTRIBUTION PRINCIPLE] *Let $A \in \mathcal{B}(\mathbb{R}^n)$ and $\alpha \in [0,n]$. If there are finite constants $c, R > 0$ and a mass distribution μ on A such that*

$$\sup_{x\in A} \mu(B[x,r]) \leq c\, r^\alpha \qquad \forall r \in (0,R],$$

then $\mathcal{H}^\alpha(A) \geq c^{-1}\mu(A) > 0$ and, in particular, $\dim A \geq \alpha$.

Proposition 2.7 is standard, see, for instance, Theorem 1.4 of [Mör03]. The *closed support* $\mathrm{supp}(\mu)$ of a Borel measure μ on $\mathbb{R}^n$ is defined to be the smallest closed set $F \subset \mathbb{R}^n$ with $\mu(\mathbb{R}^n \setminus F) = 0$. In particular, $\mathrm{supp}(\mu) \in \mathcal{B}(\mathbb{R}^n)$. The following result is an immediate consequence of Proposition 2.7 and, if $\mu(\mathbb{R}^n) = \infty$, of Proposition 2.6($ii$).

Proposition 2.8 *Let μ be a Borel measure on $\mathbb{R}^n$ and $\alpha \in [0,n]$. If the condition*

$$\exists c, R > 0: \qquad \sup_{x\in\mathbb{R}^n} \mu(B[x,r]) \leq c\, r^\alpha \qquad \forall r \in (0,R] \tag{2.2}$$

is satisfied, then $\dim \mathrm{supp}(\mu) \geq \alpha$.

Remark 2.9 *Lemma 4.2 below includes the following statements. If μ is a Borel measure on $\mathbb{R}^n$ satisfying (2.2), then we obtain for every $\alpha' \in (0,\alpha)$:*

$$\exists c, R > 0: \qquad \sup_{x\in\mathbb{R}^n} \int_{B[x,R]} |x-y|^{-\alpha'} \mu(dy) < \infty. \tag{2.3}$$

Conversely, if (2.3) holds for some $\alpha' \in [0,n]$, then (2.2) holds for $\alpha = \alpha'$.

Clearly, the closed support of the Lebesgue measure in $\mathbb{R}^n$ has Hausdorff dimension n whereas the closed support of any Dirac measure $\delta_c(dx)$, $c \in \mathbb{R}^n$, has Hausdorff dimension 0. In the next section we give an example for a Borel measure on $\mathbb{R}^n$ having closed support with Hausdorff dimension strictly between 0 and n.

2.5 Example: Cantor set and measure

Fix $\lambda \in (0, 1/2)$. We are going to construct the λ-Cantor set on the interval $I_{0,1} := [0,1]$. First cut out the middle part of $I_{0,1}$ in such a manner that the intervals $I_{1,1} := [0,\lambda]$ and $I_{1,2} := [1-\lambda, 1]$ remain. Next cut out the middle parts of $I_{1,1}$ and $I_{1,2}$ in the same proportions, i.e. the intervals $I_{2,1} := [0,\lambda^2]$, $I_{2,2} := [(1-\lambda)\lambda, \lambda]$, $I_{2,3} := [1-\lambda, 1-\lambda+\lambda^2]$ and $I_{2,4} := [1-\lambda^2, 1]$ remain. Then go ahead in this manner. More precisely, if we have already defined the intervals $I_{k-1,1}, \dots, I_{k-1,2^{k-1}}$, we define $I_{k,1}, \dots, I_{k,2^k}$ by deleting from the middle of each $I_{k-1,i}$ an interval of the length $(1-2\lambda)\lambda^{k-1}$. Each interval $I_{k,i}$ has length λ^k. The "limit set" $C(\lambda) := \cap_{k=0}^{\infty} \cup_{i=1}^{2^k} I_{k,i}$ is called λ-*Cantor set.* The set $C(\lambda)$ consists of uncountably many singletons (i.e. no point is an interior point) and is compact. To see that the number of points is uncountable, note that $C(\lambda)$ corresponds one-to-one to the uncountable set $\{0;1\}^{\mathbb{N}}$. In fact each point $x \in C(\lambda)$ can be encoded by a sequence $(r_k(x))_{k=1}^{\infty} \in \{0;1\}^{\mathbb{N}}$, where $r_k(x)$ is 0 or 1 (resp.) depending on whether x is contained in the subset of $I_{k-1,i}$ which is left or right (resp.) from the cut out of $I_{k-1,i}$, and $I_{k-1,i}$ is the unique set from $I_{k-1,1}, \dots, I_{k-1,2^{k-1}}$ that contains x.

Let us focus on the Hausdorff dimension of the λ-Cantor set $C(\lambda)$. Assume $\alpha \in [0,1]$ satisfies $0 < \mathcal{H}^{\alpha}(C(\lambda)) < \infty$, i.e. $\alpha = \dim C(\lambda)$, where $\mathcal{H}^{\alpha}$ is the α-dimensional Hausdorff measure on $\mathbb{R}$. Consequently, since $C(\lambda) = \lambda C(\lambda) \cup [(1-\lambda) + \lambda C(\lambda)]$ holds, assertions (iii) and (iv) of Proposition 2.4 yield

$$\begin{aligned}&\mathcal{H}^{\alpha}(C(\lambda)) = \mathcal{H}^{\alpha}(\lambda C(\lambda) \cup [(1-\lambda) + \lambda C(\lambda)]) = \\ &\qquad \mathcal{H}^{\alpha}(\lambda C(\lambda)) + \mathcal{H}^{\alpha}((1-\lambda) + \lambda C(\lambda)) = \lambda^{\alpha}\mathcal{H}^{\alpha}(C(\lambda)) + \lambda^{\alpha}\mathcal{H}^{\alpha}(C(\lambda)) = 2\lambda^{\alpha}\mathcal{H}^{\alpha}(C(\lambda))\end{aligned}$$

and so $\alpha = \log 2/|\log\lambda|$. In order to identify $\alpha(\lambda) := \log 2/|\log\lambda|$ with the Hausdorff dimension of $C(\lambda)$, one only has to check yet that the assumption $0 < \mathcal{H}^{\alpha(\lambda)}(C(\lambda)) < \infty$ is correct. However, Hausdorff showed $\mathcal{H}^{\alpha(\lambda)}(C(\lambda)) = 1$ (cf. [Hau19]), i.e. $\dim C(\lambda) = \alpha(\lambda)$ holds indeed. Note that $\alpha(\lambda) \in (0,1)$ since $\lambda \in (0,1/2)$. The Borel measure $\mathcal{C}_{\lambda}(.) := \mathcal{H}^{\alpha(\lambda)}(. \cap C(\lambda))$ is called λ-*Cantor measure* on $\mathbb{R}$. To some extent, $\mathcal{C}_{\lambda}$ is the uniform measure on $C(\lambda)$. It is also singular w.r.t. the Lebesgue measure $\mathcal{L}^1$ since $\mathcal{L}^1(C(\lambda)) = 0$ but $\mathcal{H}^{\alpha(\lambda)}(C(\lambda)) = 1$. Here $\mathcal{L}^1(C(\lambda)) = 0$ follows from $\alpha(\lambda) < 1$, $\mathcal{H}^{\alpha(\lambda)}(C(\lambda)) = 1$ and Proposition 2.4(i) and (ii). By the definition of $\mathcal{C}_{\lambda}$ we have $\operatorname{supp}(\mathcal{C}_{\lambda}) = C(\lambda)$, and therefore $\dim \operatorname{supp}(\mathcal{C}_{\lambda}) = \alpha(\lambda)$.

In order to define $\mathcal{C}_{\lambda}$'s analogue on $\mathbb{R}^n$ set $C(\lambda)^n := C(\lambda) \times \cdots \times C(\lambda) \in \mathcal{B}(\mathbb{R}^n)$, $\alpha_n(\lambda) := \log(2^n)/|\log\lambda|$ and let $\mathcal{H}^{\alpha_n(\lambda)}$ be the $\alpha_n(\lambda)$-dimensional Hausdorff measure on $\mathbb{R}^n$. Since $C(\lambda)^n$ is a self-similar set, one can use techniques as in Chapter 4.13 of [Mat95] to show $0 < \mathcal{H}^{\alpha_n(\lambda)}(C(\lambda)^n) < \infty$. In particular, $\dim C(\lambda)^n = \alpha_n(\lambda)$. We call $\mathcal{C}_{\lambda}^n(.) := \mathcal{H}^{\alpha_n(\lambda)}(. \cap C(\lambda)^n)$ the λ-*Cantor measure* on $\mathbb{R}^n$. Its closed support equals $C(\lambda)^n$ and it is singular w.r.t. the Lebesgue measure $\mathcal{L}^n$.

By means of Theorem 4.14 of [Mat95] one can also show that there exist finite constants $0 < c < C$ and $0 < c_n < C_n$ such that

- $c\, r^{\alpha(\lambda)} \le \sup_{x \in C(\lambda)} \mathcal{C}_{\lambda}(B[x,r]) \le C\, r^{\alpha(\lambda)} \qquad \forall r \in (0,1]$,
- $c_n\, r^{\alpha_n(\lambda)} \le \sup_{x \in C(\lambda)^n} \mathcal{C}_{\lambda}^n(B[x,r]) \le C_n\, r^{\alpha_n(\lambda)} \qquad \forall r \in (0,1]$.

In particular, $\mathcal{C}_{\lambda}$ and $\mathcal{C}_{\lambda}^n$ fulfill (2.2) with $\alpha = \alpha(\lambda)$ and $\alpha = \alpha_n(\lambda)$, respectively.

2.6 Vague and weak topology

A topological space is called *Hausdorff space* if for every distinct points s and s' there are neighborhoods V_s and $V_{s'}$ of s and s', respectively, with $V_s \cap V_{s'} = \emptyset$. A topological space is said to be *locally compact* if it is a Hausdorff space and if every point possesses at least one compact neighborhood. Assume S to be a locally compact space and introduce the following class of Borel measures μ on S:

$$\mathcal{M}(S) := \{\mu : \mu(K) < \infty \ \forall \text{ compact } K \in \mathcal{B}(S)\} \qquad (\text{“Radon measures”}).$$

We equip $\mathcal{M}(S)$ with the *vague topology*, defined as the coarsest topology on $\mathcal{M}(S)$ w.r.t. which each of the mappings $\bar{\pi}_\psi$, $\psi \in C_c(S)$, is continuous. Here $\bar{\pi}_\psi : \mathcal{M}(S) \to \mathbb{R}$ is given by $\bar{\pi}_\psi(\mu) := \langle \mu, \psi \rangle$. In particular, if $B(x, \epsilon)$ denotes the open ball in $\mathbb{R}$ around x with radius $\epsilon > 0$,

$$\mathcal{S}_v := \left\{ \bar{\pi}_\psi^{-1}\Big(B(x,\epsilon)\Big) \ : \ \psi \in C_c(S), x \in \mathbb{R}, \epsilon > 0 \right\}$$

is a subbasis of the vague topology. Then it is not hard to show that, for every $\mu \in \mathcal{M}(S)$,

$$\begin{aligned} \mathcal{V}_v(\mu) &:= \left\{ \cap_{j=1}^m \bar{\pi}_{\psi_j}^{-1}\Big(B(\bar{\pi}_{\psi_j}(\mu), \epsilon)\Big) \ : \ m \geq 1, \psi_j \in C_c(S), \epsilon > 0 \right\} \\ &= \left\{ \left\{ \nu \in \mathcal{M}(S) : |\langle \nu, \psi_j \rangle - \langle \mu, \psi_j \rangle| < \epsilon \ \ \forall j \leq m \right\} \ : \ m \geq 1, \psi_j \in C_c(S), \epsilon > 0 \right\} \end{aligned}$$

provides a basis (consisting of vaguely open sets) for μ's neighborhoods.[2] It follows easily that a sequence $(\mu_n) \subset \mathcal{M}(S)$ converges to some $\mu \in \mathcal{M}(S)$ in the vague topology if and only if $\langle \mu_n, \psi \rangle \to \langle \mu, \psi \rangle$ for all $\psi \in C_c(S)$. In this case we say (μ_n) *converges vaguely* to μ. The shape of $\mathcal{V}_v(\mu)$ also ensures that $\mathcal{M}(S)$ equipped with the vague topology is a Hausdorff space, i.e. it separates points. In fact, if $\mu \neq \mu'$, then there exists by Riesz' representation theorem (cf. [Bau92] Sätze 29.1, 29.3) some $\psi \in C_c(S)$ with $\langle \mu, \psi \rangle \neq \langle \mu', \psi \rangle$, and so one can choose $V_\mu \in \mathcal{V}_v(\mu)$ and $V_{\mu'} \in \mathcal{V}_v(\mu')$ such that $V_\mu \cap V_{\mu'} = \emptyset$. A subset H of $C_c(S)$ is called *vague convergence determining* in $\mathcal{M}(S)$ if $\langle \mu_n, \psi \rangle \to \langle \mu, \psi \rangle \ \forall \psi \in H$ implies vague convergence of (μ_n) to μ. Also, a subset H of $C_c(S)$ is said to be *separating* in $\mathcal{M}(S)$ if $\langle \mu, \psi \rangle = \langle \mu', \psi \rangle \ \forall \psi \in H$ implies $\mu = \mu'$. Riesz' representation theorem ensures that $C_c(S)$ is separating in $\mathcal{M}(S)$ and so is any vague convergence determining set. In particular, the weak limit is unique.

Recall that a topological space with countable basis is said to be *Polish* if there exists a complete metric which generates its topology. In particular, any complete and separable metric space is Polish[3]. If S is Polish, then $\mathcal{M}(S)$ is Polish, too. Indeed,

Proposition 2.10 [Polish space, vague topology] *If S is a locally compact Polish space, then one can find a countable dense subset $\{f_k\} \equiv \{f_k\}_{k \geq 1}$ of $(C_c(S), \|.\|_\infty)$ such that*

$$d_{\mathcal{M}(S)}(\mu, \mu') := \sum_{k=1}^{\infty} 2^{-k} \Big(1 \wedge |\langle \mu, f_k \rangle - \langle \mu', f_k \rangle|\Big) \tag{2.4}$$

provides a complete metric on $\mathcal{M}(S)$ which induces the vague topology. Moreover, the countable set of discrete measures $\sum_{i=1}^m w_i \delta_{s_i}(ds)$, $w_i \in \mathbb{Q}_+$ and s_i from some fixed countable dense subset of S, is dense in $(\mathcal{M}(S), d_{\mathcal{M}(S)})$. In particular, $\{f_k\}$ is vague convergence determining in $\mathcal{M}(S)$ and $\mathcal{M}(S)$ equipped with the vague topology is Polish.

[2] Note that the vague topology can alternatively be defined as the topology having $\mathcal{V}_v := \cup_{\mu \in \mathcal{M}(S)} \mathcal{V}_v(\mu)$ as system of basic neighborhoods. That is, $G \subset \mathcal{M}(S)$ is defined to be open if $\forall \mu \in G \ \exists V \in \mathcal{V}_v(\mu)$: $V \subset G$, and $\mathcal{B}_v := \mathcal{V}_v \cup \{\emptyset\}$ provides a topological basis. This is true since (1) $\mathcal{V}_v(\mu) \neq \emptyset \ \forall \mu$ and $\mu \in V \ \forall V \in \mathcal{V}_v(\mu)$, (2) $V_1, V_2 \in \mathcal{V}_v(\mu) \Rightarrow \exists V \in \mathcal{V}_v(\mu)$: $V \subset V_1 \cap V_2$ and (3) $\forall V \in \mathcal{V}_v(\mu) \ \forall \nu \in V \ \exists U \in \mathcal{V}_v(\nu)$: $U \subset V$.

[3] For a metrizable space, separability and the existence of a countable topological basis are equivalent (cf. [Fra73] Satz 10.7).

Proof See, for instance, the statement and the proof of Satz 31.5 of [Bau92]. $\square$

Let us give a word on the construction of the countable set $\{f_k\}$ yet. Any locally compact space S with countable topological basis is σ-compact (see [Bau92] p.209), i.e. there exists a sequence (K_n) of compact sets such that $K_n \uparrow S$ and every compact set K is contained in finally all K_n. By Urysohn's lemma there exist functions $e_n \in C_c(S)$, $n \geq 1$, such that $0 \leq e_n \leq 1$ and $e_n = 1$ on K_n. If $\{\tilde{f}_l\}$ denotes any countable dense subset of $(C_c(S), \|.\|_\infty)$, then the set $\{f_k\}$ in Proposition 2.10 can be defined as $\{f_k\}_{k\geq 1} := \{\tilde{f}_l\}_{l\geq 1} \cup \{\tilde{f}_l e_n\}_{l\geq 1, n\geq 1} \cup \{e_n\}_{n\geq 1}$. Next we wish to characterize the Borel σ-algebra in $\mathcal{M}(S)$. Let $\mathcal{A}(S)$ denote the algebra of all relatively compact sets from $\mathcal{B}(S)$ and define the mapping $\bar{\pi}_A : \mathcal{M}(S) \to [0, \infty)$ by $\bar{\pi}_A(\mu) := \mu(A)$.

Proposition 2.11 [Borel σ-algebra, vague topology] *If S is a locally compact Polish space, then $\mathcal{B}(\mathcal{M}(S))$, i.e. the σ-algebra generated by the vaguely open sets, is generated by the functions $\bar{\pi}_\psi$, $\psi \in C_c(S)$, and also by the functions $\bar{\pi}_A$, $A \in \mathcal{A}(S)$.*

Proof Let $\mathcal{T}_v$ denote the vague topology and let $\mathcal{B}_v$ be its basis defined in the footnote of page 12. According to Proposition 2.10, there exists a countable basis $\mathcal{B}_v^0$ of the vague topology. We may assume[4] $\mathcal{B}_v^0 \subset \mathcal{B}_v$. Every vaguely open set, i.e. every set from $\mathcal{T}_v$, is hence a countable union of sets from $\mathcal{B}_v$, in particular it is contained in $\sigma(\mathcal{B}_v)$. Thus $\mathcal{B}(\mathcal{M}(S)) \stackrel{\text{def}}{=} \sigma(\mathcal{T}_v) \subset \sigma(\mathcal{B}_v)$. Also, the system $\mathcal{B}_v$ is contained in $\sigma(\bar{\pi}_\psi^{-1}(\mathcal{B}(\mathbb{R})) : \psi \in C_c(S))$ which implies $\mathcal{B}(\mathcal{M}(S)) \subset \sigma(\bar{\pi}_\psi : \psi \in C_c(S))$. On the other hand, every map $\bar{\pi}_\psi$ is vaguely continuous, i.e. in particular, $\bar{\pi}_\psi^{-1}(B(x, \epsilon)) \in \mathcal{T}_v$ $(\subset \mathcal{B}(\mathcal{M}(S))$ for every $x \in \mathbb{R}$ and $\epsilon > 0$. However, $\{B(x, \epsilon) : x \in \mathbb{R}, \epsilon > 0\}$ is a generating system for $\mathcal{B}(\mathbb{R})$. Thus a standard argument yields $\bar{\pi}_\psi^{-1}(\mathcal{B}(\mathbb{R})) \subset \mathcal{B}(\mathcal{M}(S))$ and so $\sigma(\bar{\pi}_\psi : \psi \in C_c(S)) \subset \mathcal{B}(\mathcal{M}(S))$. Hence we have $\sigma(\bar{\pi}_\psi : \psi \in C_c(S)) = \mathcal{B}(\mathcal{M}(S))$ which was the first claim.

For the proof of the second claim it is helpful to take the following fact into account. The vague topology can alternatively be defined as the coarsest topology w.r.t. which each of the mappings $\bar{\pi}_A$, $A \in \mathcal{A}(S)$, is continuous (cf. Lemma 1.4 of [Kal83]). Then $\sigma(\bar{\pi}_A : A \in \mathcal{A}(S)) = \mathcal{B}(\mathcal{M}(S))$ follows as above. $\square$

We now assume S to be a metric space and we introduce the following classes of Borel measures[5] μ on S:

$$
\begin{aligned}
\mathcal{M}_f(S) &:= \{\mu : \mu(S) < \infty\} \qquad (\text{“finite measures”}) \\
\mathcal{M}_1(S) &:= \{\mu : \mu(S) = 1\} \qquad (\text{“probability measures”}).
\end{aligned}
$$

If S is also locally compact, then these classes are of course subclasses of $\mathcal{M}(S)$. We can make $\mathcal{M}_f(S)$ a measurable space by furnishing it with the *weak topology*, i.e. with the coarsest topology on $\mathcal{M}_f(S)$ w.r.t. which each of the mappings $\bar{\pi}_\psi$, $\psi \in C_b(S)$, is continuous. As in the case of vague convergence we can show that a sequence $(\mu_n) \subset \mathcal{M}(S)$ converges to some $\mu \in \mathcal{M}(S)$ in the weak topology if and only if $\langle \mu_n, \psi \rangle \to \langle \mu, \psi \rangle$ for

[4] If a topological space S possesses a countable topological basis, then for every topological basis $\mathcal{B}$ of S there is a countable topological basis $\mathcal{B}^0$ of S such that $\mathcal{B}^0 \subset \mathcal{B}$, cf. [Ale64] p.189.

[5] If $[S, \mathcal{S}]$ is only a measurable space, then $\mathcal{M}_f(S)$ and $\mathcal{M}_1(S)$ refer to the classes of measures μ on $[S, \mathcal{S}]$ satisfying $\mu(S) < \infty$, respectively $\mu(S) = 1$.

all $\psi \in C_b(S)$. In this case we say (μ_n) *converges weakly* to μ. We keep the notion of convergence determining and separating sets. However, the role of $C_c(S)$ is played by $C_b(S)$. In particular, we speak of *weak convergence determining sets*, respectively of *separating sets in* $\mathcal{M}_f(S)$. The space $C_b(S)$ is separating in $\mathcal{M}_f(S)$ (since S is metric, cf. [Bau92] Lemma 30.14) and so is any weak convergence determining set. In particular, the weak limit is unique. If S is also locally compact, then $C_c(S)$ is already separating in $\mathcal{M}_f(S)$ since it is separating in $\mathcal{M}(S)$ ($\supset \mathcal{M}_f(S)$). Further (cf. [Par67], Theorems II.6.2 and II.6.5),

Proposition 2.12 [POLISH SPACE, WEAK TOPOLOGY] *If S is a complete and separable metric space, then $\mathcal{M}_f(S)$ equipped with the weak topology is a Polish space.*

That is, provided S is complete and separable, $\mathcal{M}_f(S)$ equipped with the weak topology possesses a complete and separable metrization. In particular, any metric which induces the weak topology is complete.

Remark 2.13 *If S is a complete and separable metric space, then one can find a countable subset $\{g_k\} \equiv \{g_k\}_{k\geq 1}$ of $C_b^+(S)$ such that $\mathbf{1} \in \{g_k\}$ and $\{g_k\}$ is weak convergence determining in $\mathcal{M}_f(S)$ (cf. [Daw93], Lemma 3.2.1). In particular,*[6]

$$d_{\mathcal{M}_f(S)}(\mu, \mu') := \sum_{k=1}^{\infty} 2^{-k} \Big(1 \wedge |\langle \mu, g_k\rangle - \langle \mu', g_k\rangle|\Big) \tag{2.5}$$

provides a (complete) metric on $\mathcal{M}_f(S)$ which induces the weak topology.

We can make the subclass $\mathcal{M}_1(S)$ of $\mathcal{M}_f(S)$ a measurable space by furnishing it with the induced weak topology. In particular, $\mathcal{B}(\mathcal{M}_1(S)) = \mathcal{B}(\mathcal{M}_f(S)) \cap \mathcal{M}_1(S)$. The following analogue of Proposition 2.11 can be proved as Proposition 2.11.

Proposition 2.14 [BOREL σ-ALGEBRA, WEAK TOPOLOGY] *If S is a complete and separable metric space, then $\mathcal{B}(\mathcal{M}_f(S))$, i.e. the σ-algebra generated by the weakly open sets, is generated by the functions $\bar{\pi}_\psi$, $\psi \in C_b(S)$, and also by the functions $\bar{\pi}_B$, $B \in \mathcal{B}(S)$.*

Clearly, if S is locally compact, then every weakly convergent sequence $(\mu_n) \subset \mathcal{M}(S)$ is also vaguely convergent. The converse holds under some additional assumption. In fact, a sequence $(\mu_n) \subset \mathcal{M}_f(S)$ converges weakly to $\mu \in \mathcal{M}_f(S)$ if (μ_n) converges vaguely to μ and $\lim_{n\to\infty}\langle \mu_n, \mathbf{1}\rangle = \langle \mu, \mathbf{1}\rangle$ (cf. [Bau92] Satz 30.8). In particular, a sequence $(\mu_n) \subset \mathcal{M}_1(S)$ converges weakly to $\mu \in \mathcal{M}_1(S)$ if and only if it converges vaguely to μ. Note, however, that a vaguely convergent sequence $(\mu_n) \subset \mathcal{M}_1(S)$ does not need to converge to a probability measure, i.e. the vague limit μ does not need to lie in $\mathcal{M}_1(S)$. In fact, the sequence may loose mass, i.e. $\langle \mu, \mathbf{1}\rangle < 1$ is possible (cf. [Bau92] p.218 Beispiel 2 and its continuation on p.221). The loosing of mass can be ruled out if the vaguely convergent sequence $(\mu_n) \subset \mathcal{M}_1(S)$ is known to be tight. In this case the vague limit does lie in $\mathcal{M}_1(S)$, in

[6] $d_{\mathcal{M}_f(S)}$ is clearly a metric on $\mathcal{M}_f(S)$. Also, it is easy to see that a sequence (μ_n) converges weakly to μ if and only if it converges to μ w.r.t. $d_{\mathcal{M}_f(S)}$. In particular, since the weak topology is metrizable, $d_{\mathcal{M}_f(S)}$ induces the weak topology.

particular (μ_n) converges weakly to μ, (cf. [Bau92] p.227, Bemerkung 3). Here a sequence $(\mu_n) \subset \mathcal{M}_1(S)$ is said to be *tight* if:

$$\forall \epsilon > 0 \;\; \exists \, \text{compact } K \subset S : \qquad \mu_n[K] \geq 1 - \epsilon \quad \forall n \geq 1.$$

The property of tightness on its own already implies relative compactness w.r.t. weak convergence (cf. Section 1.6 of [Bil68]):

Theorem 2.15 [PROHOROV'S THEOREM] *Let S be a metric space. Then every tight sequence $(\mu_n) \subset \mathcal{M}_1(S)$ is relatively compact in $\mathcal{M}_1(S)$ w.r.t. weak convergence. If the metric space S is assumed to be separable and complete, then the converse is also true.*

Concluding this section we state yet the following two simple but basic lemmas:

Lemma 2.16 *Let S be a metric space and $\mu, \mu_1, \mu_2, \ldots \in \mathcal{M}_1(S)$. Then (μ_n) converges weakly to μ if and only if for every sequence $(\mu_{n'}) \subset (\mu_n)$ there exists a subsequence $(\mu_{n''}) \subset (\mu_{n'})$ such that $(\mu_{n''})$ converges weakly to μ.*

Lemma 2.17 *Let S, S' be metric spaces and $h : S \to S'$ be continuous. Then weak convergence of $(\mu_n) \subset \mathcal{M}_1(S)$ to $\mu \in \mathcal{M}_1(S)$ implies weak convergence of $(\mathbb{P}_n \circ h^{-1}) \subset \mathcal{M}(S')$ to $\mathbb{P} \circ h^{-1} \in \mathcal{M}_1(S')$.*

2.7 Kernels

This section is devoted to the definition of kernels from a measurable space $(\Omega, \mathcal{F})$ to a measurable space $(S, \mathcal{S})$. Let $[0, \infty] = [0, \infty) \cup \{\infty\}$ denote the Alexandrov compactification of $[0, \infty)$. Hence $\mathcal{B}([0, \infty]) = \sigma(\mathcal{G})$ where $\mathcal{G}$ consists of all open subsets of $[0, \infty)$ and all sets from $\{[0, \infty] \setminus K : K \subset [0, \infty) \text{ compact}\}$. Note that $\mathcal{B}([0, \infty]) = \sigma(\mathcal{B}([0, \infty)) \cup \{\infty\})$.

Definition 2.18 [KERNEL] *A map $\xi : \Omega \times \mathcal{S} \to [0, \infty]$ is called kernel from Ω to S if*

(i) $\omega \mapsto \xi(\omega, B)$ *is* $[\mathcal{F}, \mathcal{B}([0, \infty])]$*-measurable,* $\quad \forall B \in \mathcal{S}$

(ii) $\xi(\omega, .)$ *is a measure on* $(S, \mathcal{S})$*,* $\quad \forall \omega \in \Omega$.

If we require in *(ii)* that $\xi(\omega, .)$ is even an element of $\mathcal{M}(S)$, $\mathcal{M}_f(S)$ or $\mathcal{M}_1(S)$, then ξ is said to be a *Radon-*, *finite-* or *probability kernel*, respectively. When considering Radon measures, we tacitly assume that S is a locally compact space and $\mathcal{S} = \mathcal{B}(S)$. Let $\mathfrak{M}(S)$ and $\mathfrak{M}_f(S)$[7] denote the σ-algebras on $\mathcal{M}(S)$ and $\mathcal{M}_f(S)$, respectively, generated by the functions $\bar{\pi}_B$, $B \in \mathcal{S}$. Note that $\bar{\pi}_B$ on $\mathcal{M}(S)$ may take the value ∞ if B is not compact, i.e. $\bar{\pi}_B : \mathcal{M}(S) \to [0, \infty]$. In view of the following lemma, a Radon kernel (resp. finite kernel) can alternatively be seen as a measurable mapping from $(\Omega, \mathcal{F})$ to $(\mathcal{M}(S), \mathfrak{M}(S))$ (resp. $(\mathcal{M}_f(S), \mathfrak{M}_f(S))$).

Lemma 2.19 *A family $\xi = \{\xi(\omega, .) : \omega \in \Omega\}$ of Radon measures on S is a Radon kernel from Ω to S if and only if the map $\xi : \Omega \to \mathcal{M}(S)$, $\omega \mapsto \xi(\omega, .)$ is $[\mathcal{F}, \mathfrak{M}(S)]$-measurable. Analogously, a family $\xi = \{\xi(\omega, .) : \omega \in \Omega\}$ of finite measures on S is a finite kernel from Ω to S if and only if the map $\xi : \Omega \to \mathcal{M}_f(S)$, $\omega \mapsto \xi(\omega, .)$ is $[\mathcal{F}, \mathfrak{M}_f(S)]$-measurable.*

[7] If S is a complete and separable metric space, then $\mathfrak{M}_f(S)$ coincides with the Borel σ-algebra (w.r.t. the weak topology) in $\mathcal{M}_f(S)$, cf. Proposition 2.14.

Proof We only prove the statement on Radon kernels, the statement on finite kernels can be shown analogously. If $\xi : \omega \mapsto \xi(\omega, .)$ is $[\mathcal{F}, \mathfrak{M}(S)]$-measurable, then $\xi : \omega \mapsto \xi(\omega, B)$ is $[\mathcal{F}, \mathcal{B}([0,\infty])]$-measurable (and hence a Radon kernel) since $\bar{\pi}_B : \mu \mapsto \mu(B)$ is $[\mathfrak{M}(S), \mathcal{B}([0,\infty])]$-measurable. Conversely, let ξ be a Radon kernel. Then $\omega \mapsto \xi(\omega, B)$ is $[\mathcal{F}, \mathcal{B}([0,\infty])]$-measurable and we obtain for every $B \in \mathcal{S}$ and $H \in \mathcal{B}([0,\infty])$

$$\xi^{-1}(\bar{\pi}_B^{-1}(H)) = \{\omega \in \Omega : \bar{\pi}_B(\xi(\omega, .)) \in H\} = \{\omega \in \Omega : \xi(\omega, B) \in H\} = \xi^{-1}(., B)(H) \in \mathcal{F}.$$

Since $\{\bar{\pi}_B^{-1}(H) : B \in \mathcal{S}, H \in \mathcal{B}([0,\infty])\}$ is a generating system for $\mathfrak{M}(S)$, we can conclude $[\mathcal{F}, \mathfrak{M}(S)]$-measurability of $\xi : \omega \mapsto \xi(\omega, .)$. □

Remark 2.20 *If μ_1 and μ_2 are two (Radon-, finite-, probability-) kernels from S to S, then $\mu_1\mu_2(s, ds'') := \mu_2(s', ds'')\mu_1(s, ds')$ is a (Radon-, finite-, probability-) kernel from S to S, too (see, for instance, [Kal83] p.20).*

2.8 Condition (A) and condition (B)

Concluding this chapter we introduce two classes of Borel measures on $[0,\infty) \times \mathbb{R}$, respectively $[0,\infty) \times \mathbb{R}^d$, which play a central role in this thesis. Recall that $B[x, r]$ denotes the closed ball around x with radius r.

Definition 2.21 [CONDITION (A)] *We say a Borel measure $\mu(dtdx) = \mu_1(t, dx)\mu_2(dt)$ on $[0,\infty) \times \mathbb{R}$ satisfies condition (A) if μ_1 is a kernel from $[0,\infty)$ to $\mathbb{R}$, $\mu_2(dt)$ is a Borel measure on $[0,\infty)$ and if there are $\alpha_1 \in [0,1]$, $\alpha_2 \in [0,1]$ such that:*

(i) $\forall T > 0\ \exists c_T > 0 : \quad \sup_{t \le T} \sup_{x \in \mathbb{R}} \mu_1(t, B[x, r]) \le c_T\, r^{\alpha_1} \quad \forall r \in (0,1]$,

(ii) $\forall T > 0\ \exists c_T > 0 : \quad \sup_{t \le T} \mu_2([0,\infty) \cap B[t, r]) \le c_T\, r^{\alpha_2} \quad \forall r \in (0,1]$,

(iii) $\frac{\alpha_1}{2} + \alpha_2 > 1$.

Definition 2.22 [CONDITION (B)] *We say a Borel measure $\mu(dtdx) = \mu_1(t, dx)\mu_2(dt)$ on $[0,\infty) \times \mathbb{R}^d$ satisfies condition (B) if μ_1 is a kernel from $[0,\infty)$ to $\mathbb{R}^d$, $\mu_2(dt)$ is a Borel measure on $[0,\infty)$ and if there are $\beta_1 \in [0,d]$, $\beta_2 \in [0,1]$ such that:*

(i) $\forall T > 0\ \exists c_T > 0 : \quad \sup_{t \le T} \sup_{x \in \mathbb{R}^d} \mu_1(t, B[x, r]) \le c_T\, r^{\beta_1} \quad \forall r \in (0,1]$,

(ii) $\forall T > 0\ \exists c_T > 0 : \quad \sup_{t \le T} \mu_2([0,\infty) \cap B[t, r]) \le c_T\, r^{\beta_2} \quad \forall r \in (0,1]$,

(iii) $\frac{\beta_1}{2} + \beta_2 > \frac{d}{2}$.

Note that condition (A) and condition (B) can be reformulated by means of Remark 2.9. If a Borel measure $\mu(dtdx) = \mu_1(t, dx)\mu_2(dt)$ satisfies one of the conditions (A) or (B), then μ_2 has to be a Radon measure and μ_1 needs to take values in $\mathcal{M}_{uni}(\mathbb{R})$, respectively $\mathcal{M}_{uni}(\mathbb{R}^d)$, where:

$$\mathcal{M}_{uni}(\mathbb{R}^d) := \{\mu \text{ Borel measure on } \mathbb{R}^d \ : \ \sup_{x \in \mathbb{R}^d} \mu(B[x, 1]) < \infty\} \subset \mathcal{M}(\mathbb{R}^d).$$

Condition (B) is clearly weaker than condition (A). Also, condition (A) requires α_1 and α_2 to be strictly positive whereas condition (B) allows β_1 to be 0 when $d = 1$ and $\beta_2 > 1/2$. In particular, in the latter case $\mu_1(t, dx)$ may have spatial atoms. Recall from Proposition 2.8 that the exponents $\alpha_1, \alpha_2, \beta_1$ and β_2 provide lower bounds for the Hausdorff dimension of the closed supports of the corresponding Borel measures. Let us give some examples.

Let $\mathcal{C}_\lambda(dx)$ denote the λ-Cantor measure on $\mathbb{R}$ $(0 < \lambda < \frac{1}{2})$ which was introduced in Section 2.5. Then $\mu(dtdx) = dtdx$ and $\mu(dtdx) = \mathcal{C}_\lambda(dx)dt$ satisfy condition (A). Furthermore, $\mu(dtdx) = \mathcal{C}_{\lambda_1}(dx)\mathcal{C}_{\lambda_2}(dt)$ satisfies condition (A) for any $\lambda_1, \lambda_2 \in (0, \frac{1}{2})$ with

$$\log 2/|2 \log \lambda_1| + \log 2/|\log \lambda_2| > 1.$$

The Lebesgue measures $\mu(dtdx) = dtdx$ on $[0, \infty) \times \mathbb{R}^d$ trivially satisfies condition (B) for every $d \geq 1$. In Section 2.5 we also introduced the λ-Cantor measure $\mathcal{C}^d_\lambda(dx)$ on $\mathbb{R}^d$ $(d \geq 1$, $0 < \lambda < \frac{1}{2})$. The Borel measures $\mu(dtdx) = \mathcal{C}^d_\lambda(dx)dt$ and $\mu(dtdx) = \mathcal{C}^d_{\lambda_1}(dx)\mathcal{C}_{\lambda_2}(dt)$ satisfy condition (B) if $\lambda, \lambda_1, \lambda_2 \in (0, \frac{1}{2})$ such that

$$\log(2d)/|2 \log \lambda| + 1 > \frac{d}{2}, \quad \text{respectively} \quad \log(2d)/|2 \log \lambda_1| + \log 2/|\log \lambda_2| > \frac{d}{2}.$$

As already mentioned, in dimension $d = 1$ a Borel measure $\mu(dtdx)$ satisfying condition (B) may have spatial atoms. For instance, $\mu(dtdx) = \delta_0(dx)dt$ and even $\mu(dtdx) = \delta_0(dx)\mathcal{C}_{\lambda_2}(dt)$ (with $|\log 2|/|\log \lambda_2| > \frac{1}{2}$) satisfy condition (B).

3 Foundations of stochastic processes

This chapter concerns foundations of stochastic processes. The first section recalls basic definitions and a few basic results. In Sections 3.2 and 3.3 we focus on path regularity of processes with complete metric state space E and index set $I = \mathbb{R}^m$, $m \geq 1$. In particular we consider the cases $E = C_{tem}(\mathbb{R}^d)$ and $E = C_{rap}(\mathbb{R}^d)$. Sections 3.4 and 3.5 are devoted to tightness and weak convergence of processes, especially of $C_{tem}(\mathbb{R}^d)$-valued continuous processes. In Sections 3.6 - 3.11 we study certain types of stochastic processes that will be dealt with throughout this thesis. That are, Gaussian processes, (local) martingales, random measures, Markov processes, in particular measure-valued Markov processes.

3.1 Definitions and basics

Consider a probability space $[\Omega, \mathcal{F}, \mathbb{P}]$, a measurable *state space* $[E, \mathcal{E}]$ and an abstract *index set* I. We write E^I for the class of functions $f : I \to E$, and let $\mathcal{E}^{I_0}$ denote the σ-algebra in E^I generated by the projections $\pi_t : E^I \to E$, $t \in I_0 \subset I$, given by $\pi_t(f) := f(t)$.

Definition 3.1 [STOCHASTIC PROCESS] *An $[\mathcal{F}, \mathcal{E}^I]$-measurable function $X : \Omega \to E^I$ is called an E-valued (stochastic) process on $[\Omega, \mathcal{F}, \mathbb{P}]$ with index set I. For every $\omega \in \Omega$, $X(\omega)$ is called a path or a sample of X.*

If X is an E-valued process with index set I, then $X_t = \pi_t \circ X$ maps Ω into E for every $t \in I$. Thus, X may also be regarded as a function $(t, \omega) \mapsto X_t(\omega)$ from $I \times \Omega$ to E. The following Lemma (cf. [Kal97], p.24) shows that Definition 3.1 is equivalent to regarding an E-valued process X as a collection $(X_t : t \in I)$ of random elements in the space E.

Lemma 3.2 *A function $X : \Omega \to E^I$ is $[\mathcal{F}, \mathcal{E}^I]$-measurable if and only if $X_t : \Omega \to E$ is $[\mathcal{F}, \mathcal{E}]$-measurable for every $t \in I$.*

The probability measure $\mathbb{P}_X := \mathbb{P} \circ X^{-1}$ on $[E^I, \mathcal{E}^I]$ is called the *law* or the *distribution* of a process X. We say two processes are *versions* of each other if they have the same law. If $k \geq 1$ and $t_1, \ldots, t_k \in I$, then the probability measure $\mathbb{P} \circ (X_{t_1}, \ldots, X_{t_k})^{-1}$ on E^k is said to be a *finite-dimensional distribution* of the process X. The class of all finite-dimensional distributions of a process X determines the law of X. This follows from

Theorem 3.3 [LAW AND FINITE-DIMENSIONAL DISTRIBUTIONS] *Let $I_0 \subset I$ such that $\mathcal{E}^I = \sigma(\pi_t : t \in I_0)$. The laws of two E-valued processes X and X' coincide if and only if $\mathbb{P} \circ (X_{t_1}, \ldots, X_{t_k})^{-1} = \mathbb{P} \circ (X'_{t_1}, \ldots, X'_{t_k})^{-1}$ holds for all $k \geq 1$ and $t_1, \ldots, t_k \in I_0$.*

Proof We can follow the lines of the proof of Proposition 2.2 of [Kal97] for $I_0 = I$. □

Note that two processes which are versions of each other, i.e. two processes with the same law, might be defined on different probability spaces. Checking equality of the finite-dimensional distribution, i.e. equality of the laws, is only one possibility to identify two stochastic processes. There are further – and in fact stronger – notions of equivalence of two processes. Let X and X' be two E-valued processes on the same probability space. Then X and X' are said to be *modifications* of each other if $\mathbb{P}[X_t = X'_t] = 1$ for all

$t \in I$. They are called *indistinguishable* if $\mathbb{P}$-almost surely: $X_t = X'_t$ for all $t \in I$. Clearly, indistinguishable processes are modifications of each other, and any modification X' of a process X is also a version of X.

Now, consider an E-valued process X and recall that $\mathbb{P}_X$ denotes its law. The mappings $\tilde{X}_t := \pi_t : E^I \to E$, $t \in I$, are $[\mathcal{E}^I, \mathcal{E}]$-measurable by the definition of $\mathcal{E}^I$. According to Lemma 3.2 they hence form an E-valued process $\tilde{X}$ on $[E^I, \mathcal{E}^I, \mathbb{P}_X]$. Clearly,

$$\mathbb{P} \circ X^{-1} (= \mathbb{P} \circ X^{-1} \circ \tilde{X}^{-1}) = \mathbb{P}_X \circ \tilde{X}^{-1}.$$

That is, X and $\tilde{X}$ have the same law. We call the process $\tilde{X}$ the *canonical version* of X on the *canonical path space* E^I. One also refers to $\tilde{X}$ as the *coordinate process* of $\mathbb{P}_X$.

A family $(\mathcal{F}_t) \equiv (\mathcal{F}_t)_{t \geq 0}$ of σ-algebras $\mathcal{F}_t \subset \mathcal{F}$ is called *filtration* in $\mathcal{F}$ if $\mathcal{F}_t \subset \mathcal{F}_{t+s}$ holds for all $s, t \geq 0$. Intuitively $\mathcal{F}_t$ is the information known to an observer at time t. A random variable τ with values in $[0, \infty]$ is called $(\mathcal{F}_t)$-stopping time if $\{\omega : \tau(\omega) \leq t\} \in \mathcal{F}_t$ for every $t \geq 0$. For a process $X = (X_t : t \geq 0)$ with index set $I = [0, \infty)$ we set $\mathcal{F}^X_{I_0} := \sigma(X_r : r \in I_0)$ for every $I_0 \subset I$ as well as $\mathcal{F}^X_t := \mathcal{F}^X_{[0,t]}$ for every $t \geq 0$. The families $(\mathcal{F}^X_{[s,t]}) \equiv (\mathcal{F}^X_{[s,t]})_{t \geq s}$ and $(\mathcal{F}^X_t) \equiv (\mathcal{F}^X_t)_{t \geq 0}$ are easily seen to be filtrations; they are called the *natural filtrations induced by* X. $\mathcal{F}^X_{[s,t]}$ represents the information obtained by observing X during the interval $[s, t]$. A process $X = (X_t : t \geq 0)$ is said to be *adapted* to a filtration $(\mathcal{F}_t)$ if X_t is $\mathcal{F}_t$-measurable for each $t \geq 0$. Trivially, a process $X = (X_t : t \geq 0)$ is adapted to the filtration $(\mathcal{F}^X_t)$. A process $(X_t : t \geq 0)$ is said to be *progressively measurable* w.r.t. a filtration $(\mathcal{F}_t)$ if the mapping $[0, t] \times \Omega \to E, (r, \omega) \mapsto X_r(\omega)$ is $\mathcal{B}([0, t]) \times \mathcal{F}_t$-measurable. Any $(\mathcal{F}_t)$-progressively measurable process is also $(\mathcal{F}_t)$-adapted. Occasionally we need additional structure on $(\mathcal{F}_t)$. A filtration $(\mathcal{F}_t)$ is said to be $\mathbb{P}$-*complete* if $[\Omega, \mathcal{F}, \mathbb{P}]$ is complete and $N_{\mathbb{P}} \subset \mathcal{F}_0$. It is said to be *right-continuous* if $\mathcal{F}_t = \mathcal{F}_{t+}$ for all $t \geq 0$, where $\mathcal{F}_{t+} := \cap_{\epsilon > 0} \mathcal{F}_{t+\epsilon}$. We say $(\mathcal{F}_t)$ satisfies the *usual conditions* if it is both $\mathbb{P}$-complete and right-continuous. An arbitrary filtration $(\mathcal{F}_t)$ can always be completed: one first completes the probability space (let the completion be denoted by $[\Omega, \tilde{\mathcal{F}}^{\mathbb{P}}, \tilde{\mathbb{P}}]$) and then defines $\tilde{\mathcal{F}}^{\mathbb{P}}_t := \sigma(\mathcal{F}_t \cup N_{\tilde{\mathbb{P}}})$, where $N_{\tilde{\mathbb{P}}}$ denotes the class of $\tilde{\mathbb{P}}$-null set in $\tilde{\mathcal{F}}^{\mathbb{P}}$. The filtration $(\tilde{\mathcal{F}}^{\mathbb{P}}_t)$ is said to be the $\mathbb{P}$-*completion* of $(\mathcal{F}_t)$. The filtration $(\bar{\mathcal{F}}^{\mathbb{P}}_t) :\equiv (\tilde{\mathcal{F}}^{\mathbb{P}}_{t+})$ is called the *usual augmentation* of $(\mathcal{F}_t)$ w.r.t. $\mathbb{P}$ and satisfies the usual conditions. If there is no risk of ambiguity, we suppress the superscript $\mathbb{P}$ and write $(\bar{\mathcal{F}}_t)$ instead of $(\bar{\mathcal{F}}^{\mathbb{P}}_t)$.

Remark 3.4 *Any modification of an $(\mathcal{F}_t)$-adapted process is also $(\mathcal{F}_t)$-adapted, provided $\mathcal{F}_0$ contains all the $\mathbb{P}$-null sets in $\mathcal{F}$ (in particular if $(\mathcal{F}_t)$ satisfies the usual conditions).*

A real-valued process $(X_t : t \in I)$ with index set I is said to be *uniformly integrable* if

$$\lim_{r \to \infty} \sup_{t \in I} \mathbb{E}[|X_t| \, \mathbf{1}_{|X_t| > r}] = 0. \tag{3.1}$$

More generally, a family $X = ((\Omega_t, \mathcal{F}_t, \mathbb{P}_t, X_t) : t \in I)$ of real-valued random variables is said to be *uniformly integrable* if (3.1) with $\mathbb{E}$ replaced by $\mathbb{E}_t$ holds. The following Lemma (cf. [Kal97] p.44) provides a sufficient condition for a family to be uniformly integrable.

Lemma 3.5 *A family $X = ((\Omega_t, \mathcal{F}_t, \mathbb{P}_t, X_t) : t \in I)$ of real-valued random variables is uniformly integrable if it is L^p-bounded, i.e. $\sup_{t \in I} \mathbb{E}_t[|X_t|^p] < \infty$, for some $p > 1$.*

3.2 Sample continuity of processes

In this section we assume a metric structure d on the state space E, i.e. $\mathcal{E}$ may and will be chosen as the Borel σ-algebra $\mathcal{B}(E)$. Let the index set I be $\mathbb{R}^m$, $m \geq 1$. Definition 3.1 does not provide any detailed information about the behavior of the process' samples. In many cases one wishes the samples of a process $X = (X_t : t \in I)$ to be continuous, i.e. $(X_t(\omega) : t \in I) \in C(I, E)$ for every $\omega \in \Omega$. From a technical point of view it is often sufficient to know that the samples are only $\mathbb{P}$-almost surely continuous, that means $(X_t(\omega) : t \in I) \in C(I, E)$ for $\mathbb{P}$-almost all $\omega \in \Omega$[8]. Already in that case a process is said to be *continuous*. In fact there is no significant restriction since any continuous process X possesses a modification X' whose samples are all continuous. If Ω_0 denotes the exceptional null set, then X' can be obtained by setting $X'_t(\omega) := X_t(\omega)$ for $\omega \notin \Omega_0$ and $X'_t(\omega) := x_0$ for $\omega \in \Omega_0$, for every $t \in I$, where x_0 is an arbitrary fixed element of E. At this point we stress the fact that continuity of a process does not mean $\mathbb{P}_X[C(I, E)] = 1$. In fact, the latter expression does not make any sense since $C(I, E) = \{f \in E^I : t \mapsto \pi_t(f) \text{ continuous}\} \notin \mathcal{E}^I$. Roughly speaking, a subset of E^I can only lie in $\mathcal{E}^I$ if it depends on only countably many coordinates. In other words, the σ-algebra $\mathcal{E}^I$ is "too small" for a space as big as E^I. For details see [Bil95] p.493. The following result (cf. [Wal86] Corollary 1.2, or [Kal97] Theorem 2.23) provides a convenient tool for checking whether a process possesses a continuous modification.

Proposition 3.6 [Kolmogorov's continuity criterion] *Assume (E, d) is complete and let $X = (X_t : t \in \mathbb{R}^m)$ be an E-valued process with index set $I = \mathbb{R}^m$. Then X has a continuous modification X' if there exist finite constants $\epsilon, q > 0$ such that*

$$\exists R > 0,\ c_R > 0 : \qquad \mathbb{E}[d(X_t, X_{t'})^q] \leq c_R |t - t'|^{m+\epsilon} \quad \forall t, t' \in \mathbb{R}^m : \ |t - t'| \leq R.$$

In this case, X' is locally Hölder-γ-continuous for every $\gamma \in (0, \frac{\epsilon}{q})$. In fact, for every compact $K \subset \mathbb{R}^m$ there exist a non-negative random variable M and a finite constant $c = c_{q,m,\epsilon,\gamma} > 0$ so that $\mathbb{E}[M^q] \leq c$ and $d(X'_t, X'_{t'}) \leq M|t - t'|^\gamma$ $\forall t, t' \in K$, $\mathbb{P}$-almost surely.

For further analysis we furnish $C(I, E)$ with the σ-algebra $\bar{\mathcal{E}}^I$ generated by the maps $\bar{\pi}_t$, $t \in I$, where $\bar{\pi}_t$ is the restriction of π_t to $C(I, E)$. Note that $\bar{\mathcal{E}}^I = \mathcal{E}^I \cap C(I, E)$ holds. However, $\bar{\mathcal{E}}^I \not\subset \mathcal{E}^I$ since $C(I, E) \notin \mathcal{E}^I$. We have seen that any continuous process X possesses a modification X' such that $X'_\cdot(\omega) \in C(I, E)$ for all $\omega \in \Omega$, i.e. $X' : \Omega \to C(I, E)$. This does not necessarily mean that X' is a random element in $[C(I, E), \bar{\mathcal{E}}^I]$ since X' is only known to be $[\mathcal{F}, \mathcal{E}^I]$-measurable but $\bar{\mathcal{E}}^I \not\subset \mathcal{E}^I$. However, we can construct a version of X', and in fact of any continuous process X, which may be identified with a random element in $[C(I, E), \bar{\mathcal{E}}^I]$: If X is a continuous process on the basic probability space $[\Omega, \mathcal{F}, \mathbb{P}]$, then we can unambiguously define a probability measure $\bar{\mathbb{P}}$ on $[C(I, E), \bar{\mathcal{E}}^I]$ by setting $\bar{\mathbb{P}}[\bar{H}] := \mathbb{P}_X[H]$ for any set $H \in \mathcal{E}^I$ with $\bar{H} = H \cap C(I, E)$. By Lemma 3.2, $\bar{X} = (\bar{X}_t := \bar{\pi}_t : t \in I)$ provides an E-valued process[9] on $[C(I, E), \bar{\mathcal{E}}^I, \bar{\mathbb{P}}]$ since the maps $\bar{X}_t$, $t \in I$, are $[\bar{\mathcal{E}}^I, \mathcal{E}]$-measurable by the definition of $\bar{\mathcal{E}}^I$. $\bar{X}$ is called *coordinate process*

[8] i.e. there exists some $\Omega_0 \in \mathcal{F}$ such that $\mathbb{P}[\Omega_0] = 0$ and $X_\cdot(\omega)$ is continuous for all $\omega \notin \Omega_0$.

[9] In particular, $\bar{X} : C(I, E) \to E^I$ is $[\bar{\mathcal{E}}^I, \mathcal{E}^I]$-measurable.

of the probability measure $\bar{\mathbb{P}}$ and all of its samples are clearly in $C(I, E)$. If we set $\bar{H} := H \cap C(I, E)$ for $H \in \mathcal{E}^I$, then we have $\bar{X}^{-1}(H) = \bar{H}$ and so

$$\mathbb{P}_X[H] \Big(= \bar{\mathbb{P}}[\bar{H}] = \bar{\mathbb{P}}[\bar{X}^{-1}(H)] = \bar{\mathbb{P}} \circ \bar{X}^{-1}[H] \Big) = \bar{\mathbb{P}}_{\bar{X}}[H] \qquad \forall H \in \mathcal{E}^I.$$

That means X and $\bar{X}$ have the same law. Therefore one also refers to $\bar{X}$ as the *canonical version* of X on the *canonical path space* $C(I, E)$. The coordinate process $\bar{X}$ of a probability measure $\bar{\mathbb{P}}$ on $[C(I, E), \bar{\mathcal{E}}^I]$ can in particular be seen as a random element in $C(I, E)$ since $\bar{X} : C(I, E) \to C(I, E)$ is trivially $[\bar{\mathcal{E}}^I, \bar{\mathcal{E}}^I]$-measurable. We summarize:

Remark 3.7 *Every probability measure on $[C(I, E), \bar{\mathcal{E}}^I]$ induces a process with continuous samples, namely its coordinate process. Conversely, for every E-valued continuous process X on $[\Omega, \mathcal{F}, \mathbb{P}]$ with index set $I = \mathbb{R}^m$ there exists a probability measure $\bar{\mathbb{P}}$ on $[C(I, E), \bar{\mathcal{E}}^I]$ whose coordinate process $\bar{X}$ has the same law as X. Also, the coordinate process of any probability measure on $[C(I, E), \bar{\mathcal{E}}^I]$ can be identified with a random element in $C(I, E)$.*

If (E, d) is complete and separable, then $\bar{\mathcal{E}}^I$ coincides with $\mathcal{B}(C(I, E))$, where $\mathcal{B}(C(I, E))$ is the Borel σ-algebra in $C(I, E)$ w.r.t. the *topology $\mathcal{T}_\infty$ of uniform convergence on compacts* (cf. [Kal97]: Lemma 14.1 and the remark on p.259). For any metric space (E, d), the latter topology is defined to be generated by the family $\{d_K : K \subset I \text{ compact}\}$ of semi-metrics on $C(I, E)$, where $d_K(f, f') := \sup_{t \in K} d(f(t), f'(t))$. That means $\mathcal{T}_\infty$ is the (unique) topology with $\mathcal{V}_\infty := \cup_{f \in C(I,E)} \mathcal{V}_\infty(f)$ as system of basic neighborhoods, where

$$\mathcal{V}_\infty(f) := \Big\{ \Big\{ g \in C(I, E) : d_{K_j}(f, g) < \epsilon \ \ \forall j \leq m \Big\} \ : \ m \geq 1, K_j \subset I \text{ compact}, \epsilon > 0 \Big\}.$$

It follows easily that a sequence $(f_n) \subset C(I, E)$ converges to some $f \in C(I, E)$ in the topology $\mathcal{T}_\infty$ of uniform convergence on compacts if and only if $d_K(f_n, f) \to 0$ for every compact set $K \subset I$. In fact, its is enough to consider only countably many semi-metrics: The family $\{d_{K_n} : n \geq 1\}$ generates the same topology whenever $K_n \uparrow I$. In that case,

$$d_\infty(f, f') := \sum_{n=1}^{\infty} 2^{-k} \ \big[1 \wedge d_{K_n}(f, f')\big] \tag{3.2}$$

provides a metric on $C(I, E)$ which generates $\mathcal{T}_\infty$, too. Note that $(C(I, E), \mathcal{T}_\infty)$ is Polish if (E, d) is complete and separable (cf. [Bau92], Satz 31.6).

3.3 Processes with samples in $C([0, \infty), C_{tem}(\mathbb{R}^d))$

Let $X = (X_t(x) : t \geq 0, x \in \mathbb{R}^d)$ be a real-valued process with index set $I = [0, \infty) \times \mathbb{R}^d$, $d \geq 1$. We are going to study the following question: When can $(X_t(.) : t \geq 0)$ be assumed to be $C_{tem}(\mathbb{R}^d)$-valued (or $C_{rap}(\mathbb{R}^d)$-valued) continuous, where

$$\begin{aligned} C_{tem}(\mathbb{R}^d) &:= \{f \in C(\mathbb{R}^d) \ : \ |f|_{(\lambda)} < \infty \ \ \forall \lambda < 0\} \\ C_{rap}(\mathbb{R}^d) &:= \{f \in C(\mathbb{R}^d) \ : \ |f|_{(\lambda)} < \infty \ \ \forall \lambda > 0\} \end{aligned}$$

and $|f|_{(\lambda)} := \|f(.) e^{\lambda |.|}\|_\infty$. Here $C_{tem}(\mathbb{R}^d)$ (resp. $C_{rap}(\mathbb{R}^d)$) is assumed to be furnished with the topology $\mathcal{T}_{tem}$ (resp. $\mathcal{T}_{rap}$) generated by the family of seminorms $\{|.|_{(\lambda)} : \lambda < 0\}$ (resp.

$\{|.|_{(\lambda)} : \lambda < 0\})$[10]. In fact, it is enough to consider only countably many seminorms: the family $\{|.|_{(-1/k)} : k \geq 1\}$ (resp. $\{|.|_{(k)} : k \geq 1\}$) generates the same topology. Also,

$$d_{tem}(f, f') := \sum_{k=1}^{\infty} 2^{-k} \left[1 \wedge |f - f'|_{(-1/k)}\right] \; \Big(\text{resp. } d_{rap}(f, f') := \sum_{k=1}^{\infty} 2^{-k} \left[1 \wedge |f - f'|_{(k)}\right] \Big)$$

provides a metric on $C_{tem}(\mathbb{R}^d)$ (resp. $C_{rap}(\mathbb{R}^d)$) which generates $\mathcal{T}_{tem}$ (resp. $\mathcal{T}_{rap}$), too. $(C_{tem}(\mathbb{R}^d), \mathcal{T}_{tem})$ and $(C_{rap}(\mathbb{R}^d), \mathcal{T}_{rap})$ are Polish spaces and, in particular, Fréchet spaces[11]. The following result can also be found in [Shi94] for the case $d = 1$, but without proof.

Proposition 3.8 [$C_{tem}(\mathbb{R}^d)$-VALUED CONTINUITY] *Let $X = (X_t(x) : t \geq 0, x \in \mathbb{R}^d)$ be a real-valued process such that $X_0(.) \in C_{tem}(\mathbb{R}^d)$ $\mathbb{P}$-almost surely. Assume there are constants $q, \epsilon > 0$ such that for every $\lambda, T > 0$ there exists a constant $c_{\lambda,T} > 0$ satisfying*

$$\mathbb{E}\Big[|X_t(x) - X_{t'}(x')|^q\Big] \leq c_{\lambda,T}\Big(|t - t'|^{1+d+\epsilon} + |x - x'|^{1+d+\epsilon}\Big)e^{\lambda|x|} \tag{3.3}$$

for all $t, t' \leq T$ and $x, x' \in \mathbb{R}^d$ with $|x - x'| \leq 1$. Then X has a modification X' such that $(X'_t(.) : t \geq 0)$ is $C_{tem}(\mathbb{R}^d)$-valued continuous. Moreover, X' is locally jointly Hölder-γ-continuous for each $\gamma \in (0, \frac{\epsilon}{q})$.

Proof For every $k, l, T \geq 1$ set

$$A_k := \{x \in \mathbb{R}^d \; : \; k - 1 \leq |x| \leq k\} \quad \text{and} \quad D_l^{k,T} := \Big([0, T] \times A_k\Big) \cap \Big(2^{-l}\mathbb{Z}\Big)^{1+d}.$$

We further define $D_l^T := \cup_{k\geq 1} D_l^{k,T}$ and $D^T := \cup_{l\geq 1} D_l^T$ as well as

$$M_{l,\lambda}^{k,T} := \sup\Big\{|X_t(x) - X_{t'}(x')|e^{-\lambda(k-1)} \; : \; (t, x), (t', x') \in D_l^{k,T} \text{ with } |(t, x) - (t', x')| = 2^{-l}\Big\}.$$

Each set

$$\Big\{((t, x), (t', x')) \in D_l^{k,T} \times D_l^{k,T} \; : \; |(t, x) - (t', x')| = 2^{-l}\Big\}$$

consists of less than $\{[T(2k)^d][2^{1+d}2(1 + d)2^{(1+d)l}]\}$ elements. Hence we obtain by (3.3) for every $\gamma \in (0, \frac{\epsilon}{q})$ and $\lambda > 0$:

$$\begin{aligned}
&\mathbb{E}\Big[\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \Big(2^{\gamma l} M_{l,\lambda}^{k,T}\Big)^q\Big] \\
&\quad \leq \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \mathbb{E}\Big[\Big(2^{\gamma l} M_{l,\lambda/2}^{k,T} e^{-(\lambda/2)(k-1)}\Big)^q\Big]
\end{aligned}$$

[10] $\mathcal{T}_{tem}$ and $\mathcal{T}_{rap}$ are generated by $\{|.|_{(\lambda)} : \lambda < 0\}$, respectively $\{|.|_{(\lambda)} : \lambda > 0\}$, in the same way as $\mathcal{T}_\infty$ was generated by $\{d_K : K \subset I \text{ compact}\}$, cf. Section 3.2.

[11] A topological vector space $(S, \mathcal{T})$ is said to be pre-Fréchet space if its topology $\mathcal{T}$ is generated by a countable family of seminorms and if $\mathcal{T}$ has the Hausdorff property. It is called Fréchet space if it is also complete. Note that a topological vector space generated by a countable family of seminorms has the Hausdorff property if and only if 0 is the only vector for which all seminorms vanish. For a brief discussion of (pre-)Fréchet spaces see [Jän99] Section 2.5.

$$
\begin{aligned}
&= \sum_{k=1}^{\infty} e^{-q(\lambda/2)(k-1)} \sum_{l=1}^{\infty} 2^{q\gamma l} \mathbb{E}\Big[\Big(M_{l,\lambda/2}^{k,T}\Big)^q\Big] \\
&\leq \sum_{k=1}^{\infty} e^{-q(\lambda/2)(k-1)} \sum_{l=1}^{\infty} 2^{q\gamma l} \Big\{[T(2k)^d][2^{1+d}2(1+d)2^{(1+d)l}]\Big\} c_{q(\lambda/2),T}\, e^{+q(\lambda/2)}\, 2^{-l(1+d+\epsilon)} \\
&\leq \sum_{k=1}^{\infty} e^{-q(\lambda/2)(k-1)} \Big(T(2k)^d 2^{1+d} 2(1+d)\Big)\, e^{+q(\lambda/2)}\, c_{q(\lambda/2),T} \sum_{l=1}^{\infty} 2^{(q\gamma-\epsilon)l} \\
&\leq c_{q(\lambda/2),T,(q\gamma-\epsilon),d} \sum_{k=1}^{\infty} e^{-q(\lambda/2)(k-1)}\, k^d \;<\; \infty.
\end{aligned}
$$

Consequently, there exists some $\Omega_0 \in \mathcal{F}$ such that $\mathbb{P}[\Omega_0] = 0$ and for every $\omega \notin \Omega_0$:

$$
\exists\, \tilde{c}_{\lambda,T}(\omega) > 0 : \qquad M_{l,\lambda}^{k,T}(\omega) \leq \tilde{c}_{\lambda,T}(\omega)\, 2^{-\gamma l} \qquad \forall k, l \geq 1. \tag{3.4}
$$

Two points (t,x), $(t',x') \in D^T$ with $|(t,x)-(t',x')| \leq 2^{-m}$ can be connected by a walk on D^T involving, for each $l \geq m$, at most $2(1+d)$ steps between nearest neighbors in D_l^T. Thus, if $m(h)$ is the unique integer m with $2^{-(m+1)} < h \leq 2^{-m}$, we obtain by (3.4):

$$
\begin{aligned}
&\sup\Big\{|X_t(x) - X_{t'}(x')|e^{-\lambda|x|} \;:\; (t,x),(t',x') \in D^T \text{ with } |(t,x)-(t',x')| \leq h\Big\} \\
&\quad\leq \sum_{l=m(h)}^{\infty} 2(1+d)\tilde{c}_{\lambda,T} 2^{-\gamma l} \;\leq\; 2(1+d)\tilde{c}_{\lambda,T} \sum_{j=0}^{\infty} 2^{-\gamma(m(h)+j)} \\
&\quad= 2(1+d)\tilde{c}_{\lambda,T}\Big(\sum_{j=0}^{\infty} 2^{-\gamma(j-1)}\Big) 2^{-\gamma(m(h)+1)} \;\leq\; \bar{c}_{\lambda,T} 2^{-\gamma(m(h)+1)} \;<\; \bar{c}_{\lambda,T} h^{\gamma}
\end{aligned}
$$

for all $h \in (0,1]$, $\mathbb{P}$-almost surely (i.e. on $\Omega \setminus \Omega_0$). In particular, $(X_t(x) : (t,x) \in D^T)$ is locally Hölder-γ-continuous on $\Omega \setminus \Omega_0$. Now set $X''(\omega) :\equiv 0$ for $\omega \in \Omega_0$, and

$$
X_t''(\omega, x) := \begin{cases} X_t(\omega, x) & , \quad (t,x) \in D^T \\ \lim_{D^T \ni (r,y) \to (t,x)} X_r(\omega, y) & , \quad \text{otherwise} \end{cases}
$$

for $\omega \notin \Omega_0$. With help of (3.3) it is easy to show that $(X_t''(x) : (t,x) \in [0,T] \times \mathbb{R}^d)$ provides a process which is a modification of $(X_t(x) : (t,x) \in [0,T] \times \mathbb{R}^d)$ and which satisfies

$$
\begin{aligned}
&\sup\Big\{|X_t''(\omega,x) - X_{t'}''(\omega,x')|e^{-\lambda|x|} : \\
&\qquad (t,x),(t',x') \in [0,T] \times \mathbb{R}^d \text{ with } |(t,x)-(t',x')| \leq h\Big\} \;\leq\; \bar{c}_{\lambda,T}(\omega) h^{\gamma}
\end{aligned} \tag{3.5}
$$

for all $h \in (0,1]$ and $\omega \in \Omega$. In particular, $(X_t''(x) : (t,x) \in [0,T] \times \mathbb{R}^d)$ is locally Hölder-γ-continuous. Now it is easy to construct a modification $X' = (X_t'(x) : t \geq 0, x \in \mathbb{R}^d)$ of $X = (X_t(x) : t \geq 0, x \in \mathbb{R}^d)$ such that (3.5) holds for all $\lambda = 1, \frac{1}{2}, \frac{1}{3}, \ldots$ and $T = 1, 2, 3, \ldots$, $\mathbb{P}$-almost surely. The process X' is clearly locally Hölder-γ-continuous, and $(X_t'(.) : t \geq 0)$ is $C_{tem}(\mathbb{R}^d)$-valued continuous (since $X_0 \in C_{tem}(\mathbb{R}^d)$ $\mathbb{P}$-almost surely). $\square$

Proposition 3.9 [$C_{rap}(\mathbb{R}^d)$-VALUED CONTINUITY] *Let $X = (X_t(x) : t \geq 0, x \in \mathbb{R}^d)$ be a real-valued process such that $X_0(.) \in C_{rap}(\mathbb{R}^d)$ $\mathbb{P}$-almost surely. Assume there are constants $q, \epsilon > 0$ such that for every $\lambda, T > 0$ there exists a constant $c_{\lambda,T} > 0$ satisfying*

$$\mathbb{E}\Big[|X_t(x) - X_{t'}(x')|^q\Big] \leq c_{\lambda,T}\Big(|t-t'|^{1+d+\epsilon} + |x-x'|^{1+d+\epsilon}\Big)e^{-\lambda|x|}$$

for all $t, t' \leq T$ and $x, x' \in \mathbb{R}^d$ with $|x - x'| \leq 1$. Then X has a modification X' such that $(X'_t(.) : t \geq 0)$ is $C_{rap}(\mathbb{R}^d)$-valued continuous. Moreover, X' is locally jointly Hölder-γ-continuous for each $\gamma \in (0, \frac{\epsilon}{q})$.

Proof The proof goes along the lines of the proof of Proposition 3.8 with slight changes. Instead of $M_{l,\lambda}^{k,T}$ consider

$$\tilde{M}_{l,\lambda}^{k,T} := \sup\Big\{|X_t(x) - X_{t'}(x')|e^{+\lambda k} \ : \ (t,x),(t',x') \in D_l^{k,T} \text{ with } |(t,x)-(t',x')| = 2^{-l}\Big\}$$

and use the inequality $\tilde{M}_{l,\lambda}^{k,T} \leq \tilde{M}_{l,2\lambda}^{k,T}e^{-\lambda k}$ instead of $M_{l,\lambda}^{k,T} \leq M_{l,\lambda/2}^{k,T}e^{-(\lambda/2)(k-1)}$. □

3.4 Weak convergence of continuous processes

In Section 2.6 we introduced the notion of weak convergence of finite Borel measures on some metric space S. In particular, we discussed the special case of probability measures. We here focus on that case. Assume S is chosen to be the function space $C(I,E)$ where $I = \mathbb{R}^m$ and E is some metric space. Then d_∞, which was defined in (3.2), imposes a metric structure on $S = C(I,E)$. We further assume E to be complete and separable whereby $\bar{\mathcal{E}}^I = \mathcal{B}(C(I,E))$, cf. the end of Section 3.2. Consider continuous E-valued stochastic processes $X, X_1, X_2, \ldots$ with index set $I = \mathbb{R}^m$. According to Remark 3.7 we can find probability measures $\mathbb{P}, \mathbb{P}_1, \mathbb{P}_2, \ldots$ on $[C(I,E), \mathcal{B}(C(I,E))]$ which can be identified with the laws of these processes. If $(\mathbb{P}_n)$ converges weakly to $\mathbb{P}$, i.e. if

$$\int \psi(f)\mathbb{P}(df) = \lim_{n\to\infty}\int \psi(f)\mathbb{P}_n(df) \qquad \forall \psi \in C_b(C(I,E))$$

holds, then the sequence (X_n) is also said to *converge weakly to* X. Note that the weak limit is unique (in law) since $C_b(C(I,E))$ is separating in $\mathcal{M}_f(C(I,E)) \supset \mathcal{M}_1(C(I,E))$.

A crucial result in the context of weak convergence of continuous processes is Prohorov's theorem (Theorem 2.15). It states that tightness of $(\mathbb{P}_n)$ implies relative compactness of $(\mathbb{P}_n)$ w.r.t. weak convergence. That means, for every subsequence $(\mathbb{P}_{n'})$ of a tight sequence $(\mathbb{P}_n)$ there exist a subsequence $(\mathbb{P}_{n''}) \subset (\mathbb{P}_{n'})$ and a probability measure $\mathbb{P}''$ such that $(\mathbb{P}_{n''})$ converges weakly to $\mathbb{P}''$. Any limit point $\mathbb{P}''$ induces in particular a continuous process, namely its coordinate process. In Section 6.5 below we will use this argument for the construction of solutions to certain stochastic partial differential equations. Although the question ≫*When is a tight sequence even weakly convergent?*≪ does not play any role for this thesis, we mention that Lemma 2.16 gives the answer: A tight sequence $(\mathbb{P}_n)$ converges weakly to some probability measure $\mathbb{P}$ if any limit point $\mathbb{P}''$ coincides with $\mathbb{P}$. Note that in many situations it is relatively easy to prove tightness but more involved to show uniqueness of the limit points.

3.5 Tightness in $C([0,\infty), C_{tem}(\mathbb{R}^d))$

In the previous section we have seen that relative compactness (w.r.t. weak convergence) of a sequence of continuous processes (resp. their laws) is implied by tightness of the sequence of their laws. So the property of tightness is of special interest. In this section we establish tightness criteria for a sequence of probability measures on $S = C([0,\infty), C_{tem}(\mathbb{R}^d))$ equipped with the metric

$$d_{tem,\infty}(f, f') := \sum_{k=1}^{\infty} \Big(1 \wedge \sup_{t \le k} d_{tem}(f(t,.), f'(t,.))\Big)$$

(cf. Propositions 3.13, 3.14 and 3.15). Note that $(C([0,\infty), C_{tem}(\mathbb{R}^d)), d_{tem,\infty})$ is complete and separable since $(C_{tem}(\mathbb{R}^d), d_{tem})$ is (cf. the end of Section 3.2). The key for the proofs of the tightness criteria is a compactness criterion of Arzelà-Ascoli type (cf. Proposition 3.12) for subsets of $C([0,\infty), C_{tem}(\mathbb{R}^d))$. The latter will be proved with help of the following classical Arzelà-Ascoli criterion (cf. [Dud89] Sec. 2.4, or [HS71] Satz 3.10):

Theorem 3.10 [Arzelà-Ascoli criterion] *Let (K, d_K) be a compact metric space, (E, d_E) be a complete metric space and d be the supremum metric on $C(K, E)$. A set $A \subset C(K, E)$ is relatively compact w.r.t. d if and only if the following assertions hold:*

(I) $\{f(\kappa) : f \in A\}$ *is relatively compact in (E, d_E) for every $\kappa \in K$,*

(II) $\lim_{h\downarrow 0} \sup_{f\in A} \sup_{\kappa,\kappa'\in K:\, d_K(\kappa,\kappa')\le h} d_E(f(\kappa), f(\kappa')) = 0$.

Before turning to the compactness criterion for subsets of $C([0,\infty), C_{tem}(\mathbb{R}^d))$ we focus on relative compactness in the metric space $(C_{tem}(\mathbb{R}^d), d_{tem})$.

Proposition 3.11 [Arzelà-Ascoli - type criterion] *A set $A \subset C_{tem}(\mathbb{R}^d)$ is relatively compact w.r.t. d_{tem} if for every $\lambda > 0$ and $k \ge 1$ the following assertions hold:*

(i) $\sup_{f\in A} \sup_{x\in\mathbb{R}^d} |f(x)|\, e^{-\lambda|x|} < \infty$,

(ii) $\lim_{h\downarrow 0} \sup_{f\in A} \sup_{x,x'\in[-k,k]^d:\, |x-x'|\le h} |f(x) - f(x')| = 0$.

Proof Assertions (i) and (ii) clearly imply assertions (I) and (II) of Theorem 3.10 for $A_k := \{f|_{[-k,k]^d} : f \in A\} \subset (C([-k,k]^d), \|.\|_\infty)$. That is, A_k is relatively compact in $(C([-k,k]^d), \|.\|_\infty)$, for every $k \ge 1$. In order to show that A is relatively compact in $(C_{tem}(\mathbb{R}^d), d_{tem})$, let (f_n) be any sequence in A. Using the relative compactness of A_k ($\forall k \ge 1$), the assumptions (i) and (ii) as well as Cantor's diagonal argument, we can find some $f_\infty \in C_{tem}(\mathbb{R}^d)$ and a subsequence $(n_l) \subset (n)$ satisfying

$$\lim_{l\to\infty} \sup_{x\in[-k,k]^d} |f_{n_l}(x) - f_\infty(x)| = 0 \qquad \forall k \ge 1. \tag{3.6}$$

Fix some arbitrary $\lambda > 0$ and let $\epsilon > 0$. By (i) we have:

$$\sup_{f\in\{f_\infty, f_1, f_2, \ldots\}} \sup_{x\in\mathbb{R}^d} |f(x)| e^{-(\lambda/2)|x|} \;\le\; c_{\lambda/2}.$$

So we can find some $k_\epsilon = k_\epsilon(\lambda, c_{\lambda/2}) \geq 1$ such that

$$\sup_{f\in\{f_\infty,f_1,f_2,\dots\}} \sup_{x\notin[-k_\epsilon,k_\epsilon]^d} |f(x)|e^{-\lambda|x|}$$
$$\leq \Big(\sup_{f\in\{f_\infty,f_1,f_2,\dots\}} \sup_{x\in\mathbb{R}^d} |f(x)|e^{-(\lambda/2)|x|}\Big) e^{-(\lambda/2)k_\epsilon} \leq c_{\lambda/2}\, e^{-(\lambda/2)k_\epsilon} < \epsilon/4.$$

In particular, $\sup_{n\geq 1}\sup_{x\notin[-k_\epsilon,k_\epsilon]^d} |f_n(x)-f_\infty(x)|e^{-\lambda|x|} < \frac{\epsilon}{2}$. Further, by (3.6) there exists some $l_{k_\epsilon,\epsilon} \geq 1$ such that $\sup_{x\in[-k_\epsilon,k_\epsilon]^d} |f_{n_l}(x) - f_\infty(x)| < \frac{\epsilon}{2}$ for all $l \geq l_{k_\epsilon,\epsilon}$. Hence, for every $\epsilon > 0$ we can find some $l_\epsilon \geq 1$ such that

$$\sup_{x\in\mathbb{R}^d} |f_{n_l}(x) - f_\infty(x)|e^{-\lambda|x|} < \epsilon \qquad \forall l \geq l_\epsilon.$$

That is, $\lim_{l\to\infty} |f_{n_l} - f_\infty|_{(-\lambda)} = 0$. Since $\lambda > 0$ was picked arbitrarily, (f_{n_l}) is a convergent subsequence of (f_n) w.r.t. d_{tem}. Hence, A is relatively compact in $(C_{tem}(\mathbb{R}^d), d_{tem})$. □

We now turn to the compactness criterion for subsets of $(C([0,\infty), C_{tem}(\mathbb{R}^d)), d_{tem,\infty})$.

Proposition 3.12 [Arzelà-Ascoli - type criterion] *A set $A \subset C([0,\infty), C_{tem}(\mathbb{R}^d))$ is relatively compact w.r.t. $d_{tem,\infty}$ if for every $\lambda, T > 0$ and $k \geq 1$ the following assertions hold:*

(i) $\sup_{f\in A}\sup_{x\in\mathbb{R}^d} |f(0,x)|\, e^{-\lambda|x|} < \infty$,

(ii) $\lim_{h\downarrow 0}\ \sup_{f\in A}\sup_{t,t'\leq T:\ |t-t'|\leq h}\sup_{x\in\mathbb{R}^d} |f(t,x) - f(t',x)|\, e^{-\lambda|x|} = 0$,

(iii) $\lim_{h\downarrow 0}\ \sup_{f\in A}\sup_{t\leq T}\sup_{x,x'\in[-k,k]^d:\ |x-x'|\leq h} |f(t,x) - f(t,x')| = 0$.

Proof Assertions (i) and (ii) imply

$$\forall\lambda > 0, t\geq 0\ \ \exists c_{\lambda,t} > 0: \qquad \sup_{f\in A}\sup_{x\in\mathbb{R}^d} |f(t,x)|e^{-\lambda|x|} \leq c_{\lambda,t} < \infty. \tag{3.7}$$

By means of (3.7), assertion (iii) and Proposition 3.11, we get relative compactness of $\{f(t,.) : f\in A\}$ in $(C_{tem}(\mathbb{R}^d), d_{tem})$, for every $t \geq 0$. In particular, we obtain (I) of Theorem 3.10 for $A_T := \{f|_{[0,T]} : f \in A\} \subset C([0,T], C_{tem}(\mathbb{R}^d))$ for every $T > 0$. Also, assertion (ii) implies

$$\lim_{h\downarrow 0}\ \sup_{f\in A}\sup_{t,t'\leq T:\ |t-t'|\leq h} d_{tem}(f(t,.), f(t',.)) = 0,$$

i.e. (II) of Theorem 3.10 for A_T. So we have (I) and (II) of Theorem 3.12 for A_T. Hence, A_T is relatively compact in $(C([0,T], C_{tem}(\mathbb{R}^d)), d_{tem,T})$, where $d_{tem,T}(f,f') := \sup_{t\in[0,T]} d_{tem}(f(t,.), f'(t,.))$. By means of Cantor's diagonal argument, it is easy to deduce relative compactness of A in $(C([0,\infty), C_{tem}(\mathbb{R}^d)), d_{tem,\infty})$. □

We are now in the position to prove a tightness criterion for a sequence of probability measures on $C([0,\infty), C_{tem}(\mathbb{R}^d))$. As usual, we make $C([0,\infty), C_{tem}(\mathbb{R}^d))$ a measurable space by furnishing it with the Borel σ-algebra (w.r.t. $d_{tem,\infty}$). Note that the coordinate maps $\bar\pi_t : C([0,\infty), C_{tem}(\mathbb{R}^d)) \to C_{tem}(\mathbb{R}^d)$ and $\bar\pi_{(t,x)} : C([0,\infty), C_{tem}(\mathbb{R}^d)) \to \mathbb{R}$ are Borel measurable since they are continuous.

Proposition 3.13 [TIGHTNESS CRITERION] *Let $X^1, X^2, \ldots$ be the coordinate processes of probability measures $\mathbb{P}_1, \mathbb{P}_2, \ldots$ on $C([0,\infty), C_{tem}(\mathbb{R}^d))$. The sequence $(\mathbb{P}_n)$ is tight if for every $\lambda, T, \delta > 0$ and $k \geq 1$ the following assertions hold:*

(i) $\lim_{H\uparrow\infty} \sup_{n\geq 1} \mathbb{P}_n\Big[\sup_{x\in\mathbb{R}^d} |X_0^n(x)|e^{-\lambda|x|} > H\Big] = 0,$

(ii) $\lim_{h\downarrow 0}\ \sup_{n\geq 1} \mathbb{P}_n\Big[\sup_{t,t'\leq T:\ |t-t'|\leq h} \sup_{x\in\mathbb{R}^d} |X_t^n(x) - X_{t'}^n(x)|e^{-\lambda|x|} > \delta\Big] = 0,$

(iii) $\lim_{h\downarrow 0}\ \sup_{n\geq 1} \mathbb{P}_n\Big[\sup_{t\leq T} \sup_{x,x'\in[-k,k]^d:\ |x-x'|\leq h} |X_t^n(x) - X_t^n(x')| > \delta\Big] = 0.$

Proof Pick $\epsilon > 0$. For every $\lambda \in \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ and $T, k \in \{1,2,3,\ldots\}$, we can choose $H = H(\lambda, T, k) > 0$ and $h_j = h_j(\lambda, T, k) > 0$ $(j \geq 1)$ so that $h_j \downarrow 0$ (as $n \to \infty$) and:

$$\begin{aligned}
\sup_{n\geq 1} \mathbb{P}_n\Big[\sup_{x\in\mathbb{R}^d} |f(0,x)|e^{-\lambda|x|} > H\Big] &\leq \frac{1}{3}\frac{\epsilon}{2^{1/\lambda+T+k}},\\
\sup_{n\geq 1} \mathbb{P}_n\Big[\sup_{t,t'\leq T:\ |t-t'|\leq h_j} \sup_{x\in\mathbb{R}^d} |f(t,x) - f(t',x)|e^{-\lambda|x|} > \frac{1}{j}\Big] &\leq \frac{1}{3}\frac{\epsilon}{2^{1/\lambda+T+k+j}},\\
\sup_{n\geq 1} \mathbb{P}_n\Big[\sup_{t\leq T} \sup_{x,x'\in[-k,k]^d:\ |x-x'|\leq h_j} |f(t,x) - f(t,x')| > \frac{1}{j}\Big] &\leq \frac{1}{3}\frac{\epsilon}{2^{1/\lambda+T+k+j}}.
\end{aligned}$$

For the same λ, T and k we define the set

$$\begin{aligned}
A_{\lambda,T,k} := \Big\{ & f \in C([0,\infty), C_{tem}(\mathbb{R}^d)) \ : \\
& \sup_{x\in\mathbb{R}^d} |f(0,x)|e^{-\lambda|x|} \leq H,\\
& \sup_{t,t'\leq T:\ |t-t'|\leq h_j} \sup_{x\in\mathbb{R}^d} |f(t,x) - f(t',x')|e^{-\lambda|x|} \leq \frac{1}{j} \quad \forall j \geq 1,\\
& \sup_{t\leq T} \sup_{x,x'\in[-k,k]^d:\ |x-x'|\leq h_j} |f(t,x) - f(t',x')| \leq \frac{1}{j} \quad \forall j \geq 1 \Big\}.
\end{aligned}$$

Also, $A := \cap_{q,T,k=1}^{\infty} A_{1/q,T,k}$ (where $1/q$ plays the role of λ). On the one hand, the closure of A is compact by Proposition 3.12. On the other hand,

$$\begin{aligned}
\mathbb{P}_n[A] &= \mathbb{P}_n\Big[\cap_{q,T,k=1}^{\infty} A_{1/q,T,k}\Big]\\
&= \mathbb{P}_n\Big[\Big(\cup_{q,T,k=1}^{\infty} A_{1/q,T,k}^c\Big)^c\Big] \geq 1 - \sum_{q,T,k=1}^{\infty} \mathbb{P}_n[A_{1/q,T,k}^c]\\
&\geq 1 - \sum_{q,T,k=1}^{\infty} \Big(\frac{1}{3}\frac{\epsilon}{2^{q+T+k}} + \frac{1}{3}\sum_{j=1}^{\infty}\frac{\epsilon}{2^{j+q+T+k}} + \frac{1}{3}\sum_{j=1}^{\infty}\frac{\epsilon}{2^{j+q+T+k}}\Big) = 1 - \epsilon
\end{aligned}$$

holds for all $n \geq 1$. Therefore, $(\mathbb{P}_n)$ is tight. □

The following result gives another tightness criteria for a sequence of probability measures on $C([0,\infty), C_{tem}(\mathbb{R}^d))$. The conditions are stronger than (i) – (iii) of Proposition 3.13. However, in many cases it is more convenient to work with Proposition 3.14. Also, under the assumption of Proposition 3.14 any limit point is even (locally) Hölder continuous.

Proposition 3.14 [TIGHTNESS CRITERION] *Let $X^1, X^2, \ldots$ be the coordinate processes of probability measures $\mathbb{P}_1, \mathbb{P}_2, \ldots$ on $C([0,\infty), C_{tem}(\mathbb{R}^d))$. Assume there are constants $\epsilon, q > 0$ such that for every $\lambda, T > 0$ there is a constant $c_{\lambda,T} > 0$ satisfying*

$$\sup_{n\geq 1} \mathbb{E}_n \Big[|X_t^n(x) - X_{t'}^n(x')|^q \Big] \leq c_{\lambda,T} \Big(|t-t'|^{1+d+\epsilon} + |x-x'|^{1+d+\epsilon} \Big) e^{\lambda|x|} \tag{3.8}$$

for all $t, t' \leq T$ and $x, x' \in \mathbb{R}^d$ with $|x-x'| \leq 1$. If in addition $X_0 \in C_{tem}(\mathbb{R}^d)$ is fixed, then $(\mathbb{P}_n)$ is tight in $C([0,\infty), C_{tem}(\mathbb{R}^d))$. Moreover, the coordinate process of any limit point $\mathbb{P}$ is locally jointly Hölder-γ-continuous for each $\gamma \in (0, \frac{\epsilon}{q})$.

Proof Step 1. We first show that $(i)-(iii)$ of Proposition 3.13 hold whereby tightness follows. Assertion (i) trivially holds since $X_0 \in C_{tem}(\mathbb{R}^d)$. To show assertions (ii) and (iii), let D^T, D_l^T and $D_l^{k,T}$ be defined as in proof of Proposition 3.8 and set $M_{l,\lambda}^{k,T}(n) :=$

$$\sup \Big\{ |X_t^n(x) - X_{t'}^n(x')| e^{-\lambda(k-1)} : (t,x),(t',x') \in D_l^{k,T} \text{ with } |(t,x)-(t',x')| = 2^{-l} \Big\}$$

for every $k, l, n, T \geq 1$ and $\lambda > 0$. Proceeding as in the proof of Proposition 3.8 one can show that for every $\gamma \in (0, \frac{\epsilon}{q})$ and $\lambda > 0$:

$$\sup_{n\geq 1} \sup_{l\geq 1} \mathbb{E}_n \Big[\Big(\sup_{k\geq 1} 2^{\gamma l} M_{l,\lambda}^{k,T}(n) \Big)^q \Big] \quad \Big(\leq \sup_{n\geq 1} \sum_{l=1}^{\infty} \sum_{k=1}^{\infty} \mathbb{E}_n \Big[\Big(2^{\gamma l} M_{l,\lambda}^{k,T}(n) \Big)^q \Big] \Big) \quad < \infty.$$

In particular, there exists a constant $\tilde{c}_{\lambda,T} > 0$ such that

$$\sup_{n\geq 1} \mathbb{E}_n \Big[\Big(\sup_{k\geq 1} M_{l,\lambda}^{k,T}(n) \Big)^q \Big] \leq \tilde{c}_{\lambda,T} \, 2^{-\gamma l q} \qquad \forall l \geq 1. \tag{3.9}$$

Recall that two points (t,x), $(t',x') \in D^T$ with $|(t,x)-(t',x')| \leq 2^{-m}$ can be connected by a walk on D^T involving, for each $l \geq m$, at most $2(1+d)$ steps between nearest neighbors in D_l^T. Thus, if $m(h)$ denotes the unique integer m with $2^{-(m+1)} < h \leq 2^{-m}$, we obtain by a version of Minkowski's inequality (see [Kal97], Lemma 1.30) and (3.9):

$$\begin{aligned}
&\sup_{n\geq 1} \mathbb{E}_n \Big[\sup \Big\{ |X_t^n(x) - X_{t'}^n(x')| e^{-\lambda|x|} : (t,x),(t',x') \in D^T, \ |(t,x)-(t',x')| \leq h \Big\}^q \Big]^{\frac{1}{q}(q\wedge 1)} \\
&\leq \ \sup_{n\geq 1} \mathbb{E}_n \Big[\Big(\sum_{l=m(h)}^{\infty} 2(1+d) \ \sup_{k\geq 1} M_{l,\lambda}^{k,T}(n) \Big)^q \Big]^{\frac{1}{q}(q\wedge 1)} \\
&\leq \ (2(1+d))^{(q\wedge 1)} \sup_{n\geq 1} \sum_{l=m(h)}^{\infty} \mathbb{E}_n \Big[\Big(\sup_{k\geq 1} M_{l,\lambda}^{k,T}(n) \Big)^q \Big]^{\frac{1}{q}(q\wedge 1)} \ \leq \ c \sum_{l=m(h)}^{\infty} \Big(\tilde{c}_{\lambda,T} 2^{-\gamma l q} \Big)^{\frac{1}{q}(q\wedge 1)} \\
&\leq \ \bar{c}'_{\lambda,T} \Big(\sum_{j=0}^{\infty} 2^{-\gamma(j-1)(q\wedge 1)} \Big) 2^{-\gamma(m(h)+1)(q\wedge 1)} \ \leq \ \bar{c}_{\lambda,T} \ 2^{-\gamma(m(h)+1)(q\wedge 1)} \ < \ \bar{c}_{\lambda,T} \ h^{\gamma(q\wedge 1)}
\end{aligned}$$

for all $h \in (0,1]$. By the continuity of the X^n and the structure of D^T we deduce

$$\sup_{n\geq 1} \mathbb{E}_n \Big[\Big(\sup_{t,t'\leq T, |t-t'|\leq h} \ \sup_{x,x'\in\mathbb{R}^d, |x-x'|\leq h} |X_t^n(x) - X_{t'}^n(x')| e^{-\lambda|x|} \Big)^q \Big] \ < \ \bar{c}'_{\lambda,T} \ h^{\gamma q}$$

for all $h \in (0,1]$. By means of Markov's inequality we infer (for any given $\lambda, T, \delta > 0$):

$$\sup_{n\geq 1} \mathbb{P}_n\Big[\sup_{t,t'\leq T:\, |t-t'|\leq h} \; \sup_{x,x'\in\mathbb{R}^d:\, |x-x'|\leq h} |X_t^n(x) - X_{t'}^n(x')| e^{-\lambda|x|} > \delta \Big] \;<\; \frac{\bar{c}'_{\lambda,T}\, h^{\gamma q}}{\delta^q}$$

for all $h \in (0,1]$. This implies (ii) and (iii) of Proposition 3.13.

Step 2. We complete the proof by verifying the statement on the Hölder continuity. Let $\mathbb{P}$ denote the limit point of any weakly convergent subsequence $(\mathbb{P}_m) \subset (\mathbb{P}_n)$ and write X for its coordinate process. Hence, by Lemma 2.17 the law of the $\mathbb{R}_+$-valued random variable $|X_t^m(x) - X_{t'}^m(x')|^{q_0}$ converges weakly to the law of $|X_t(x) - X_{t'}(x')|^{q_0}$ for every $q_0 > 0$, $t, t' \geq 0$ and $x, x' \in \mathbb{R}^d$. Since $(.) \wedge N$ is bounded and continuous on $\mathbb{R}_+$ for every $N \geq 1$, we consequently have

$$\lim_{m\to\infty} \mathbb{E}_m[|X_t^m(x) - X_{t'}^m(x')|^{q_0} \wedge N] = \mathbb{E}[|X_t(x) - X_{t'}(x')|^{q_0} \wedge N] \qquad \forall N \geq 1. \tag{3.10}$$

Further, by (3.8), $(|X_t^m(x) - X_{t'}^m(x')|^{q_0})_{m\geq 1}$ is L^p-bounded ($p := q/q_0 > 1$) for every $q_0 \in (0,q)$. Therefore, $(|X_t^m(x) - X_{t'}^m(x')|^{q_0})_{m\geq 1}$ is also uniformly integrable (by Lemma 3.5) for every $q_0 \in (0,q)$, and so

$$\begin{aligned}
&\lim_{N\uparrow\infty} \sup_{m\geq 1} \Big| \mathbb{E}_m\Big[|X_t^m(x) - X_{t'}^m(x')|^{q_0} \wedge N \Big] - \mathbb{E}_m\Big[|X_t^m(x) - X_{t'}^m(x')|^{q_0} \Big] \Big| \qquad (3.11)\\
&\quad = \lim_{N\uparrow\infty} \sup_{m\geq 1} \mathbb{E}_m\Big[|X_t^m(x) - X_{t'}^m(x')|^{q_0} \mathbf{1}_{|X_t^m(x) - X_{t'}^m(x')|^{q_0} > N} \Big] \;=\; 0
\end{aligned}$$

holds. Moreover,

$$\begin{aligned}
&\mathbb{E}\Big[|X_t(x) - X_{t'}(x')|^{q_0} \Big] \qquad (3.12)\\
&= \mathbb{E}\Big[\lim_{N\uparrow\infty} |X_t(x) - X_{t'}(x')|^{q_0} \wedge N \Big] \;\leq\; \liminf_{N\uparrow\infty} \mathbb{E}\Big[|X_t(x) - X_{t'}(x')|^{q_0} \wedge N \Big]\\
&= \liminf_{N\uparrow\infty} \lim_{m\to\infty} \mathbb{E}_m\Big[|X_t^m(x) - X_{t'}^m(x')|^{q_0} \wedge N \Big] \;\leq\; \sup_{m\geq 1} \mathbb{E}_m\Big[|X_t^m(x) - X_{t'}^m(x')|^{q_0} \Big]\\
&< \infty.
\end{aligned}$$

On the one hand, we clearly have $|X_t(x) - X_{t'}(x')|^{q_0} \wedge N \leq |X_t(x) - X_{t'}(x')|^{q_0}$ $\forall N \geq 1$. On the other hand, $|X_t(x) - X_{t'}(x')|^{q_0}$ is in $L^1(\mathbb{P})$ by (3.12). Thus, the continuity of X and the dominated convergence theorem as well as (3.10) and (3.11) imply

$$\begin{aligned}
&\mathbb{E}\Big[|X_t(x) - X_{t'}(x')|^{q_0} \Big]\\
&= \mathbb{E}\Big[\lim_{N\uparrow\infty} |X_t(x) - X_{t'}(x')|^{q_0} \wedge N \Big]\\
&= \lim_{N\uparrow\infty} \mathbb{E}\Big[|X_t(x) - X_{t'}(x')|^{q_0} \wedge N \Big]\\
&= \lim_{N\uparrow\infty} \lim_{m\to\infty} \mathbb{E}_m\Big[|X_t^m(x) - X_{t'}^m(x')|^{q_0} \wedge N \Big]\\
&= \lim_{m\to\infty} \lim_{N\uparrow\infty} \mathbb{E}_m\Big[|X_t^m(x) - X_{t'}^m(x')|^{q_0} \wedge N \Big]\\
&= \lim_{m\to\infty} \mathbb{E}_m\Big[|X_t^m(x) - X_{t'}^m(x')|^{q_0} \Big]
\end{aligned}$$

for every $q_0 \in (0, q)$. In particular, condition (3.8) with q, $1 + d + \epsilon$ and λ replaced by by $q_0 \in (0, q)$, $(q_0/q)(1 + d + \epsilon)$ and $(q_0/q)\lambda$, respectively, extends to the limit point X. Hence, the statement on the Hölder continuity follows from Proposition 3.8 (for $\gamma \in (0, \frac{\epsilon}{q})$ choose q_0 sufficiently close to q). $\square$

The following proposition contains a slight generalization of Proposition 3.14.

Proposition 3.15 [TIGHTNESS OF SUMS] *For every $n \geq 1$, let $(\Omega_n, \mathcal{F}_n, \mathbb{P}_n)$ be a probability space and X^n, Y^n be measurable mappings from Ω_n to $C([0, \infty), C_{tem}(\mathbb{R}^d))$. In particular, $\mathbb{P}_n \circ (X^n)^{-1}$ and $\mathbb{P}_n \circ (Y^n)^{-1}$ are probability measures on $C([0, \infty), C_{tem}(\mathbb{R}^d))$.*

- *(a) If $(\mathbb{P}_n \circ (X^n)^{-1})$ and $(\mathbb{P}_n \circ (Y^n)^{-1})$ are tight, then $(\mathbb{P}_n \circ (X^n + Y^n)^{-1})$ is tight, too.*
- *(b) Assume $X := X_1 = X_2 = \ldots$ is deterministic and $(\mathbb{P}_n \circ (Y^n)^{-1})$ satisfies the assumptions of Proposition 3.14. Then $(\mathbb{P}_n \circ (X + Y^n)^{-1})$ is tight and, if Z denotes the coordinate process of any limit point of $(\mathbb{P}_n \circ (X + Y^n)^{-1})$, $Z - X$ is locally jointly Hölder-γ-continuous for each $\gamma \in (0, \frac{\epsilon}{q})$.*

Proof We first prove part (a). It is enough to show $(i) - (iii)$ of Proposition 3.13 for $(\mathbb{P}_n \circ (X^n + Y^n)^{-1})$. Since $(\mathbb{P}_n \circ (X^n)^{-1})$ and $(\mathbb{P}_n \circ (Y^n)^{-1})$ are tight, they satisfy $(i) - (iii)$ of Proposition 3.13. But then, using the elementary implication

$$|a + b| > c \quad \Longrightarrow \quad |a| > \frac{c}{2} \text{ or } |b| > \frac{c}{2}, \qquad \forall a, b \in \mathbb{R},\ c > 0,$$

one can easily deduce $(i) - (iii)$ of Proposition 3.13 for $(\mathbb{P}_n \circ (X^n + Y^n)^{-1})$.

Let us now prove part (b). The tightness of $(\mathbb{P}_n \circ (X^n + Y^n)^{-1})$ follows from Proposition 3.14 and part (a). Also, as in Step 2 of the proof of Proposition 3.14 one can show

$$\begin{aligned}
&\mathbb{E}\Big[|(Z - X)_t(x) - (Z - X)_{t'}(x')|^{q_0}\Big] \\
&\quad = \lim_{m \to \infty} \mathbb{E}_m\Big[|((X + Y^m) - X)_t(x) - ((X + Y^m) - X)_{t'}(x')|^{q_0}\Big] \\
&\quad = \lim_{m \to \infty} \mathbb{E}_m\Big[|(Y_t^m(x) - Y_{t'}^m(x')|^{q_0}\Big] \ \leq\ \sup_m \mathbb{E}_m\Big[|(Y_t^m(x) - Y_{t'}^m(x')|^{q_0}\Big] \\
&\quad \leq\ c_{\lambda,T}\Big(|t - t'|^{1+d+\epsilon} + |x - x'|^{1+d+\epsilon}\Big)^{q_0/q} e^{(q_0/q)\lambda|x|}
\end{aligned}$$

for every $q_0 \in (0, q)$ and every weakly convergent subsequent $(\mathbb{P}_m) \subset (\mathbb{P}_n)$ with limit $\mathbb{P}$. In particular, (3.8) with X^n, q, $1 + d + \epsilon$ and λ replaced by Y^m, $q_0 \in (0, q)$, $(q_0/q)(1 + d + \epsilon)$ and $(q_0/q)\lambda$, respectively, extends to $Z - X$, and the statement on the Hölder continuity follows from Proposition 3.8 (for $\gamma \in (0, \frac{\epsilon}{q})$ choose q_0 sufficiently close to q). $\square$

3.6 Gaussian processes

Here the state space E is assumed to be $\mathbb{R}$. For the index set I we do not need any further assumptions, i.e. it may be an abstract set.

Definition 3.16 [Gaussian process] *A real-valued process $X = (X_t : t \in I)$ is called a Gaussian process if its finite-dimensional distributions are Gaussian, i.e. if $\sum_{i=1}^{k} \lambda_i X_{t_i}$ is normally distributed for every $k \geq 1$, $t_1, \ldots, t_k \in I$ and $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$.*

The maps $\mu : I \to \mathbb{R}, t \mapsto \mu(t) := \mathbb{E}[X_t]$ and $\Gamma : I \times I \to \mathbb{R}, (t, t') \mapsto \Gamma(t, t') := \mathbb{C}ov(X_t, X_{t'})$ are called *mean function*, respectively *covariation function*, of the Gaussian process X. Together they determine the law of X as the next result shows (cf. [Kal97], p.200).

Proposition 3.17 [uniqueness] *The laws of two Gaussian processes with the same index set I coincide if and only if their mean- and covariation functions coincide.*

Any covariation function of a Gaussian process is clearly symmetric and positive definite. Moreover, for any symmetric and positive definite $\Gamma : I \times I \to \mathbb{R}$ we can find a corresponding centered Gaussian process. Indeed (cf. [HH93] Theorem II.2.1),

Proposition 3.18 [existence] *If $\Gamma : I \times I \to \mathbb{R}$ is symmetric and positive definite, then there exists a Gaussian process $X = (X_t : t \in I)$ with mean function $\mu \equiv 0$ and covariation function Γ.*

3.7 Martingales

We here fix $E = \mathbb{R}$ and $I = [0, \infty)$. We again denote the basic probability space by $[\Omega, \mathcal{F}, \mathbb{P}]$ and let $(\mathcal{F}_t)$ be an arbitrary filtration in $\mathcal{F}$. Occasionally we shall assume that $(\mathcal{F}_t)$ satisfies the usual conditions, which were defined in Section 3.1.

Definition 3.19 [martingale] *An $(\mathcal{F}_t)$-adapted real-valued process $M = (M_t : t \geq 0)$ with $\mathbb{E}[|M_t|] < \infty$ $\forall t \geq 0$ is called an $(\mathcal{F}_t)$-martingale if $\mathbb{E}[M_{t+s}|\mathcal{F}_t] = M_t$ $\mathbb{P}$-almost surely, for every $s, t \geq 0$. We write $\mathcal{M}$ for the class of martingales M with $M_0 = 0$ $\mathbb{P}$-almost surely. The subclass of continuous martingales from $\mathcal{M}$ is denoted by $\mathcal{M}_c$.*

An important feature of a martingale M is that the $L^p(\mathbb{P})$-norm of $\sup_{s \leq t} M_s$ can be estimated by the $L^p(\mathbb{P})$-norm of M_t whenever $p > 1$ (cf. [IW89] p.33/34):

Proposition 3.20 [Doob's inequalities] *Let M be an $(\mathcal{F}_t)$-martingale. Then we obtain for all $t > 0$ and $\lambda > 0$:*

$$\begin{aligned}
\mathbb{E}\Big[\sup_{s \leq t} |M_s|^p\Big] &\leq \Big(\frac{p}{p-1}\Big)^p \, \mathbb{E}\Big[|M_t|^p\Big] \qquad (\forall p > 1) \\
\mathbb{P}\Big[\sup_{s \leq t} |M_s| \geq \lambda\Big] &\leq \lambda^{-p} \, \mathbb{E}\Big[|M_t|^p\Big] \qquad (\forall p \geq 1).
\end{aligned}$$

A crucial role in the field of stochastic calculus is played by square-integrable martingales. A right-continuous $(\mathcal{F}_t)$-martingale M satisfying $\mathbb{E}[M_t^2] < \infty$ $\forall t \geq 0$ is said to be *square-integrable*. We write $\mathcal{M}^2$ for the class of square-integrable martingales M with $M_0 = 0$ $\mathbb{P}$-almost surely. The subclass of continuous square-integrable martingales from $\mathcal{M}^2$ is denoted by $\mathcal{M}_c^2$. For $M \in \mathcal{M}^2$ set

$$\|M\|_t := \sqrt{\mathbb{E}[M_t^2]} \qquad \forall t \geq 0$$

and

$$\|M\| := \sum_{k=1}^{\infty} 2^{-k} \Big(\|M\|_k \wedge 1 \Big). \tag{3.13}$$

Then $\|.\|$ induces a metric[12] on $\mathcal{M}^2$ when identifying indistinguishable processes (cf. [KS91] p.37). By Proposition 3.20 we have in particular $\mathbb{E}[\sup_{s\leq t} M_s^2] \leq 4\|M\|_t^2$. Moreover (cf. [KS91], Proposition 1.5.23),

Proposition 3.21 [COMPLETENESS OF $\mathcal{M}^2$] *Under the metric $\|.\|$, $\mathcal{M}^2$ is a complete metric space, and $\mathcal{M}_c^2$ is a closed subspace of $\mathcal{M}^2$.*

A fundamental result on square-integrable martingales is the so-called Doob-Meyer decomposition. Let us prepare for its presentation. A real-valued process $X = (X_t : t \geq 0)$ is said to be *simple* if it is of the form

$$X_t = \xi_0 \mathbf{1}_{\{0\}}(t) + \sum_{i=0}^{\infty} \xi_i \mathbf{1}_{(t_i, t_{i+1}]}(t)$$

where $0 = t_0 < t_1 < \cdots < t_i < \cdots \to \infty$, $\exists c > 0$: $\sup_{i\geq 0} \xi_i(\omega) < c$ $\forall \omega$ and ξ_i is $\mathcal{F}_{t_i}$-measurable. The σ-algebra in $\Omega \times [0,\infty)$ generated by the simple processes will be denoted by $\mathcal{F}_{pred}$. Note that $\mathcal{F}_{pred}$ is also generated by the $(\mathcal{F}_t)$-adapted, everywhere continuous processes (cf. [RY98], Proposition IV.5.1). A real-valued process $X = (X_t : t \geq 0)$ is called $(\mathcal{F}_t)$-*predictable* if the map $(\omega, t) \mapsto X_t(\omega)$ is $\mathcal{F}_{pred}$-measurable. Also, an $(\mathcal{F}_t)$-adapted real-valued process $A = (A_t : t \geq 0)$ is called *increasing* if $A_0 = 0$ and $t \mapsto A_t$ is non-decreasing and right-continuous. The following result provides a unique decomposition of the square of an $\mathcal{M}^2$-martingale into a sum of a predictable increasing process and another martingale. For it we need to assume that $(\mathcal{F}_t)$ satisfies the usual conditions.

Theorem 3.22 [DOOB-MEYER DECOMPOSITION] *Suppose $(\mathcal{F}_t)$ satisfies the usual conditions and let $M \in \mathcal{M}^2$. Then*

$$M_t^2 = A_t + N_t, \qquad t \geq 0 \tag{3.14}$$

holds for some $(\mathcal{F}_t)$-predictable increasing process A and some mean zero right-continuous $(\mathcal{F}_t)$-martingale N. Moreover, up to indistinguishability, the decomposition (3.14) is unique. If $M \in \mathcal{M}_c^2$, then A and N are continuous.

The unique increasing process A from (3.14) is usually denoted by $\langle M \rangle$ and called *quadratic variation process* of M. In particular, $\mathbb{E}[M_t^2] = \mathbb{E}[\langle M \rangle_t]$ for all $t \geq 0$. The proof of Theorem 3.22 can be found, for instance, on page 30 of [KS91] (where one has to be aware that an increasing process is natural if and only if it is predictable, cf. [Del72]). For right-continuous martingales (in particular for elements of $\mathcal{M}^2$) the requirement that $(\mathcal{F}_t)$ has to satisfy the usual conditions is no significant restriction. One can always switch to the usual augmentation of $(\mathcal{F}_t)$ which was defined in Section 3.1. Indeed (cf. [DM80], p.75),

[12] In fact, $d_{\|.\|}(M, M') := \|M - M'\|$ provides a metric on $\mathcal{M}^2$. However, for the sake of brevity we refer to $d_{\|.\|}$ as $\|.\|$. There is no risk of ambiguity.

Proposition 3.23 [CHANGE OF FILTRATION] *Let $M = (M_t : t \geq 0)$ be an $(\mathcal{F}_t)$-martingale and suppose it is right-continuous (i.e. $\mathbb{P}$-almost all samples are right-continuous). Then M is also a martingale w.r.t. the usual augmentation $(\bar{\mathcal{F}}_t)$ of $(\mathcal{F}_t)$.*

We next introduce local martingales which are a bit more general than martingales. For a stopping time τ and a process M set $M_t^\tau := M_{t\wedge\tau}$, $t \geq 0$.

Definition 3.24 [LOCAL MARTINGALE] *An $(\mathcal{F}_t)$-adapted real-valued process $M = (M_t : t \geq 0)$ is called a local $(\mathcal{F}_t)$-martingale if there exists a sequence $(\tau_n)_{n\geq 1}$ of $(\mathcal{F}_t)$-stopping times with $\tau_n \uparrow \infty$ $(n \to \infty)$ $\mathbb{P}$-almost surely and if $M^{\tau_n} = (M_t^{\tau_n} : t \geq 0)$ is an $(\mathcal{F}_t)$-martingale for all $n \geq 1$. We write $\mathcal{M}_{loc}$ for the class of local martingales M with $M_0 = 0$ $\mathbb{P}$-almost surely. The subclass of continuous local martingales from $\mathcal{M}$ is denoted by $\mathcal{M}_{c,loc}$.*

Every martingale is clearly a local martingale. The converse, however, fails (cf. [KS91] Remark 1.5.16). As a consequence of Theorem 3.22 we obtain:

Corollary 3.25 [DOOB-MEYER FOR LOCAL MARTINGALES] *If $M \in \mathcal{M}_{c,loc}$ and $(\mathcal{F}_t)$ satisfies the usual conditions, then there exists a unique (up to indistinguishability) predictable increasing continuous process A such that A_0=0 $\mathbb{P}$-almost surely and $M^2 - A \in \mathcal{M}_{c,loc}$.*

The process A is again denoted by $\langle M\rangle$ and called *quadratic variation process* of M. The key argument for the corollary is the following. If (τ_n) is the sequence of stopping time from Definition 3.24 and if we define $\sigma_n := \inf\{t > 0 : |M_t| = n\}$ as well as $T_n := \tau_n \wedge \sigma_n$ for every $n \geq 1$, then M^{T_n} is a martingale, too. Moreover, $|M^{T_n}| \leq n$ $\mathbb{P}$-almost surely and so $M^{T_n} \in \mathcal{M}_c^2$ for every $n \geq 1$. Consequently, by Theorem 3.22 there is for every M^{T_n} a unique predictable increasing continuous process $\langle M^{T_n}\rangle$ with $(M^{T_n})^2 - \langle M^{T_n}\rangle \in \mathcal{M}_c$. From here one easily reaches the claim of Corollary 3.25. For $M \in \mathcal{M}_c^2$ the increasing processes from Theorem 3.22 and Corollary 3.25 clearly coincide. The following result is a consequence of Theorem 3.22 and Corollary 3.25 (cf. [KS91] p.31, p.36).

Corollary 3.26 *Suppose $(\mathcal{F}_t)$ satisfies the usual conditions. If $M, M' \in \mathcal{M}^2$, then there exist predictable increasing processes $A^{(1)}, A^{(2)}$ and a mean zero $(\mathcal{F}_t)$-martingale N so that*

$$M_t M_t' = A_t + N_t, \qquad t \geq 0 \tag{3.15}$$

holds for $A := A^{(1)} - A^{(2)}$. The decomposition (3.15) is unique (up to indistinguishability) and the process A is given by $A_t = \frac{1}{4}[\langle M+M'\rangle_t - \langle M-M'\rangle_t]$. If M and M' are continuous, then A and N are continuous, too. If M and M' are only in $\mathcal{M}_{c,loc}$, then (3.15) holds for some $N \in \mathcal{M}_{c,loc}$ and the decomposition is unique as before.

The process A from (3.15) is usually denoted by $\langle M, M'\rangle$ and called *covariation process* of M and M'. In particular, $\mathbb{E}[M_t M_t'] = \mathbb{E}[\langle M, M'\rangle_t]$ for all $t \geq 0$. Two martingales M and M' are said to be *orthogonal* if $MM' = (M_t M_t' : t \geq 0)$ is a martingale, too. Hence, two $\mathcal{M}^2$-martingales M and M' are orthogonal if and only if $\langle M, M'\rangle \equiv 0$ $\mathbb{P}$-almost surely. Note that $\langle .,.\rangle$ is clearly symmetric. Also, if M_1, M_2 and M' are $\mathcal{M}^2$-martingales, then we obtain for some mean zero martingales N_1, N_2 and $N_{1,2}$:

$$\begin{aligned}\langle M_1 + M_2, M'\rangle + N_{1,2} &= (M_1 + M_2)M' \\ &= M_1M' + M_2M' = \langle M_1, M'\rangle + N_1 + \langle M_2, M'\rangle + N_2.\end{aligned}$$

The uniqueness of the decomposition (3.15) yields:

Lemma 3.27 *If $M_1, M_2, M' \in \mathcal{M}^2$, then $\langle M_1 + M_2, M' \rangle = \langle M_1, M' \rangle + \langle M_2, M' \rangle$. In particular, for any $M, M' \in \mathcal{M}^2$, we have $\langle M + M' \rangle = \langle M \rangle + 2\langle M, M' \rangle + \langle M' \rangle$.*

A fundamental result on local martingales are the Burkholder-Davis-Gundy inequalities (cf. [KS91], Theorem 3.3.28) which will frequently be used in the sequel.

Theorem 3.28 [Burkholder-Davis-Gundy inequalities] *Suppose $(\mathcal{F}_t)$ satisfies the usual conditions. For every $p > 0$ there are universal finite constants $0 < c_p \leq C_p$ (depending only on p) such that*

$$c_p \mathbb{E}[\langle M \rangle_T^p] \leq \mathbb{E}[(M_T^*)^{2p}] \leq C_p \mathbb{E}[\langle M \rangle_T^p]$$

holds for all $M \in \mathcal{M}_{c,loc}$ and each $(\mathcal{F}_t)$-stopping time $T > 0$, where $M_T^* := \max_{t \leq T} |M_t|$.

3.8 Random measures

This section is devoted to the notion of random measures on some locally compact space S. A *random measure* ξ is defined to be an $[\mathcal{M}(S), \mathfrak{M}(S)]$-valued random element on the basic probability space $[\Omega, \mathcal{F}, \mathbb{P}]$. In particular, ξ's law $\mathbb{P}_\xi := \mathbb{P} \circ \xi^{-1}$ is a probability measure on $[\mathcal{M}(S), \mathfrak{M}(S)]$. Analogously, an $[\mathcal{M}_f(S), \mathfrak{M}_f(S)]$-valued random element is said to be a *finite random measure*. In this case S does not need to be locally compact; it might be any topological (or only measurable) space. The law of a finite random measure is of course a probability measure on $[\mathcal{M}_f(S), \mathfrak{M}_f(S)]$. In view of Lemma 2.19, it is equivalent to think of a random measure (resp. finite random measure) as a Radon kernel (resp. finite kernel) from Ω to S. This justifies the following equivalent definition.

Definition 3.29 [random measure] *A random measure (resp. finite random measure) on S is a Radon kernel (resp. finite kernel) ξ from Ω to S.*

Every random measure ξ can in particular be seen as a stochastic process. In fact, by Definition 3.29, $\xi(., B)$ is $[\mathcal{F}, \mathcal{B}([0,\infty])]$-measurable for every $B \in \mathcal{B}(S)$, and $\xi(\omega, .) \in \mathcal{M}(S)$ for every $\omega \in \Omega$. In view of Lemma 3.2, $\xi = (\xi(., B) : B \in \mathcal{B}(S))$ is hence a $[0,\infty]$-valued process on $[\Omega, \mathcal{F}, \mathbb{P}]$ with index set $I = \mathcal{B}(S)$ and samples in $\mathcal{M}(S)$. Conversely, every $[0,\infty]$-valued process X on some $[\Omega, \mathcal{F}, \mathbb{P}]$ with index set $I = \mathcal{B}(S)$ and ($\mathbb{P}$-almost surely) samples in $\mathcal{M}(S)$ can be regarded as a random measure. Indeed, set $E := [0,\infty]$, $I := \mathcal{B}(S)$ and note that $\mathcal{M}(S) \subset E^I$ and $\bar{\mathcal{E}}^I (:= \mathfrak{M}_f(S)) = \mathcal{E}^I \cap \mathcal{M}(S)$. Then we can unambiguously define a probability measure $\bar{\mathbb{P}}$ on $[\mathcal{M}(S), \bar{\mathcal{E}}^I]$ by setting $\bar{\mathbb{P}}[\bar{H}] := \mathbb{P}_X[H]$ for any set $H \in \mathcal{E}^I$ with $\bar{H} = H \cap \mathcal{M}(S)$. By Lemma 3.2, $\xi = (\xi(., B) := \bar{\pi}_B : B \in I)$ provides an E-valued process[13] on $[\mathcal{M}(S), \bar{\mathcal{E}}^I, \bar{\mathbb{P}}]$ since the maps $\bar{\pi}_B$, $B \in I$, are $[\bar{\mathcal{E}}^I, \mathcal{E}]$-measurable by the definition of $\bar{\mathcal{E}}^I (= \mathfrak{M}_f(S))$. Of course, *all* samples of ξ lie in $\mathcal{M}(S)$. If we set $\bar{H} := H \cap \mathcal{M}(S)$ for $H \in \mathcal{E}^I$, then we get $\xi^{-1}(H) = \bar{H}$ and so

$$\mathbb{P}_X[H] \Big(= \bar{\mathbb{P}}[\bar{H}] = \bar{\mathbb{P}}[\xi^{-1}(H)] = \bar{\mathbb{P}} \circ \xi^{-1}[H] \Big) = \bar{\mathbb{P}}_\xi[H] \qquad \forall H \in \mathcal{E}^I.$$

That means the processes ξ and X have the same law. In particular, ξ can be seen as a random element in $\mathcal{M}(S)$, i.e. as a random measure on S, since $\xi = \bar{\pi} : \mathcal{M}(S) \to \mathcal{M}(S)$ is

[13] In particular, $\xi : \mathcal{M}(S) \to E^I$ is $[\bar{\mathcal{E}}^I, \mathcal{E}^I]$-measurable.

trivially $[\bar{\mathcal{E}}^I, \bar{\mathcal{E}}^I]$-measurable. The same arguments have been used in Section 3.2 to regard a continuous process as a random element in $C(I, E)$. Completely analogously we can regard a finite random measure as a $[0,\infty)$-valued process with index set $I = \mathcal{B}(S)$ and samples in $\mathcal{M}_f(S)$, and vice versa.

The remainder of this section will be devoted to the identification and the existence of random measures. As we will mainly consider finite random measures later on, we restrict our attention to them.

Proposition 3.30 [UNIQUENESS VIA F.D. DISTRIBUTIONS] *Let $[S, \mathcal{S}]$ be a measurable space and $\mathcal{A}$ be a subsystem of $\mathcal{S}$ such that $\sigma(\pi_A : A \in \mathcal{A}) = \mathfrak{M}_f(S)$. Then the laws of two finite random measures ξ and ξ' on S coincide if and only if*

$$\mathbb{P} \circ (\xi(A_1), \dots \xi(A_k))^{-1} = \mathbb{P} \circ (\xi'(A_1), \dots \xi'(A_k))^{-1} \quad \forall A_1, \dots, A_k \in \mathcal{A}, k \geq 1.$$

Proof Regarding ξ and ξ' as stochastic processes, Theorem 3.3 yields the claim. □

If S is a complete and separable metric space and $\mathcal{A}$ is a subsystem of $\mathcal{S} := \mathcal{B}(S)$ being closed under finite intersections and containing a basis for the topology on S, then $\sigma(\pi_A : A \in \mathcal{A}) = \mathfrak{M}_f(S)$ holds (cf. Lemma 3.2.3 of [Daw93], and recall Proposition 2.14). For $S = \mathbb{R}^m$, for instance, the algebra $\mathcal{A}(\mathbb{R}^m)$ of relatively compact (i.e. bounded) sets from $\mathcal{B}(\mathbb{R}^m)$ provides such a subsystem. Another way to show uniqueness (in law) of two finite random measures is to verify that their Laplace transforms coincide. The *Laplace transform* $L_{\mathbb{P}}$ of a probability measure $\mathbb{P}$ on $[\mathcal{M}_f(S), \mathfrak{M}_f(S)]$ is defined as the map

$$\psi \mapsto L_{\mathbb{P}}(\psi) := \int e^{-\langle \nu, \psi \rangle} \mathbb{P}(d\nu), \qquad C_b^+(S) \to [0,1].$$

If $\mathbb{P}_\xi$ is the law of a finite random measure ξ, then we also refer to $L_{\mathbb{P}_\xi} =: L_\xi$ as Laplace transform of ξ. From [Daw93] (Lemma 3.2.5) we know:

Proposition 3.31 [UNIQUENESS VIA LAPLACE TRANSFORM] *If S is a complete and separable metric space, then two probability measures $\mathbb{P}$ and $\mathbb{P}'$ on $[\mathcal{M}_f(S), \mathfrak{M}_f(S)]$ coincide if and only if $L_{\mathbb{P}}(\psi) = L_{\mathbb{P}'}(\psi)$ holds $\forall \psi \in C_b^+(S)$. In particular, the laws of two finite random measures ξ and ξ' on S coincide if and only if $L_\xi(\psi) = L_{\xi'}(\psi)$ holds $\forall \psi \in C_b^+(S)$.*

Recall that a sequence $(\psi_n) \subset C_b^+(S)$ is said to be *bp-convergent* to $\psi \in C_b^+(S)$ if $\sup_{n,s} \psi_n(s) < \infty$ and $\psi_n(s) \to \psi(s)$ for all $s \in S$. Since $C_{b,+}^\infty(\mathbb{R}^m)$ is dense in $C_{b,+}(\mathbb{R}^m)$ w.r.t. *bp*-convergence[14], we easily obtain:

Corollary 3.32 *The laws of two finite random measures ξ and ξ' on $\mathbb{R}^m$ ($m \geq 1$) coincide if and only if $L_\xi(\psi) = L_{\xi'}(\psi)$ holds for all $\psi \in C_{b,+}^\infty(\mathbb{R}^m)$.*

We just have seen how to identify random measures. Next we worry about the existence of random measures with certain properties. Recall that $[\Omega, \mathcal{F}, \mathbb{P}]$ denotes the basic probability space and suppose $\Xi : B_b(S) \to B_b(\Omega)$ is a non-negative linear functional. If Ξ satisfies a certain monotonicity condition, then there exists a finite random measure ξ on S such that $\Xi\psi(\omega) = \langle \xi(\omega), \psi \rangle$ holds for $\mathbb{P}$-almost all $\omega \in \Omega$, for every $\psi \in B_b(S)$. In fact, (see [Get74] Proposition 4.1, or [Daw92] Theorem 3.2.5):

[14]Pick $f \in C_{b,+}(\mathbb{R}^m)$ and let (P_t) denote the heat semigroup. Then, for every $t > 0$, $P_t f$ is known to be in $C_{b,+}^\infty(\mathbb{R}^m)$ and one can easily show that $P_t f$ *bp*-converge to f as $t \downarrow 0$.

Proposition 3.33 [EXISTENCE, GOOD VERSION] *Assume S is a complete and separable metric space, $[\Omega, \mathcal{F}, \mathbb{P}]$ is a probability space and $\Xi : B_b(S) \to B_b(\Omega)$ satisfies:*

(i) $\Xi(\lambda\psi + \lambda'\psi') = \lambda\Xi(\psi) + \lambda'\Xi(\psi')$ *$\mathbb{P}$-almost surely,* $\quad \forall \lambda, \lambda' \in \mathbb{R}$, $\psi, \psi' \in B_b(S)$

(ii) $\psi \in B_b(S)$, $\psi \geq 0 \implies \Xi\psi \geq 0$ *$\mathbb{P}$-almost surely,*

(iii) $(\psi_k) \subset B_b(S)$*:* $0 \leq \psi_k \uparrow \psi \in B_b(S) \implies \Xi\psi_k \uparrow \Xi\psi$ *$\mathbb{P}$-almost surely.*

Then there exists a finite kernel ξ from Ω to S (i.e. a finite random measure on S) such that $\Xi\psi(\omega) = \langle \xi(\omega), \psi \rangle$ holds for $\mathbb{P}$-almost all $\omega \in \Omega$, for every $\psi \in B_b(S)$.

The next result provides a sufficient condition for a functional $L : C_b^+(S) \to [0, 1]$ to be the Laplace transform of a probability measure on $[\mathcal{M}_f(S), \mathfrak{M}_f(S)]$. Note that a functional $L : C_b^+(S) \to \mathbb{R}$ is said to be *positive definite* if $\sum_{i,j=1}^k \lambda_i\lambda_j L(\psi_i + \psi_j) \geq 0$ holds for all $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$, $\psi_1, \ldots, \psi_k \in C_b^+(S)$ and $k \geq 1$.

Proposition 3.34 [EXISTENCE, LAPLACE TRANSFORM] *Assume S is a complete and separable metric space. Then, a functional $L : C_b^+(S) \to [0, 1]$ is the Laplace transform of a probability measure on $[\mathcal{M}_f(S), \mathfrak{M}_f(S)]$ if and only if it is bp-continuous, positive definite and satisfies $L(0) = 1$.*

Proof The statement can be found in [Dyn93] (p.1211) and was proved in a greater generality in [Fit88] (Corollary (A.6)). Actually, in [Fit88] the Laplace transform is defined on $B_b^+(S)$ instead of $C_b^+(S)$. However, the proof there also works for our setting. □

3.9 Markov processes

Consider a Polish state space E and define the index set I to be $[0, \infty)$. Let $[\Omega, \mathcal{F}, \mathbb{P}]$ denote the basic probability space and let $(\mathcal{F}_t)$ be an arbitrary filtration in $\mathcal{F}$.

Definition 3.35 [MARKOV PROCESS] *An $(\mathcal{F}_t)$-adapted E-valued process $X = (X_t : t \geq 0)$ is said to be a Markov process w.r.t. $(\mathcal{F}_t)$ if for all $B \in \mathcal{B}(E)$ and $0 \leq s \leq t$:*

$$\mathbb{P}[X_t \in B|\mathcal{F}_s] = \mathbb{P}[X_t \in B|X_s] \quad \mathbb{P}\text{-almost surely.}$$

Intuitively a Markov process can not remember the past. In other words, the evolution of the process $(X_t : t > s)$ does not depend on the realization of $(X_t : t < s)$ but only on the realization of X_s. Any Markov process is clearly Markov w.r.t. $(\mathcal{F}_t^X)$. If $(\mathcal{F}_t)$ is not specified, then we tacitly assume the use of $(\mathcal{F}_t^X)$. A crucial role in the context of Markov processes is played by Markov transition functions which are defined as follows:

Definition 3.36 [MARKOV TRANSITION FUNCTION] *A family $\mu = \{\mu_{s,t} : 0 \leq s \leq t\}$ of probability kernels $\mu_{s,t}$ from E to E is called Markov transition function if:*

(i) $\mu_{s,s}(x, .) = \delta_x(.)$, $\quad \forall s \geq 0$, $x \in E$

(ii) $\mu_{s,t}(x, B) = \int \mu_{u,t}(y, B)\mu_{s,u}(x, dy)$, $\quad \forall 0 \leq s \leq u \leq t$, $x \in E$, $B \in \mathcal{B}(E)$.

Condition (ii) is the so-called *Chapman-Kolmogorov* equation. A Markov transition function μ is said to *correspond* to a Markov process X if for every $0 \leq s \leq t$:

$$\mu_{s,t}(X_s, B) = \mathbb{P}[X_t \in B | X_s] \quad \mathbb{P}\text{-almost surely}, \qquad \forall B \in \mathcal{B}(E). \tag{3.16}$$

In particular, the conditional distributions $\mathbb{P}[X_t \in .|X_s]$ have to be regular[15]. If E is a general topological space, then the conditional distributions $\mathbb{P}[X_t \in .|X_s]$ are not necessarily regular, and so a Markov process might have no corresponding Markov transition function. However, we assumed the state space E to be Polish. This guarantees the regularity of the conditional distributions (cf. [Kal97], Theorem 5.3) and in particular the existence of a corresponding Markov transition function μ (by taking the Markov property of X additionally into account). Clearly, for each Markov process X there exists exactly one corresponding Markov transition function μ. The probability measures $\mu_{s,t}(x, .)$, $0 \leq s \leq t$ and $x \in E$, are also called *transition probabilities*. The corresponding Markov transition function μ and the initial distribution $\mathbb{P} \circ X_0^{-1}$ determine the law of X. This is a consequence of Theorem 3.3 and the following result (cf. [Kal97], Proposition 7.2) which shows that μ and $\mathbb{P} \circ X_0^{-1}$ determine the finite-dimensional distributions.

Proposition 3.37 [FINITE-DIMENSIONAL DISTRIBUTIONS] *Let $X = (X_t : t \geq 0)$ be an E-valued Markov process with Markov transition function μ. Then*

$$\mathbb{P} \circ (X_{t_1}, \ldots, X_{t_k})^{-1}(.) = \int \mu_{0,t_1} \times \mu_{t_1,t_2} \times \cdots \times \mu_{t_{k-1},t_k}(x, .)\, \mathbb{P} \circ X_0^{-1}(dx) \tag{3.17}$$

holds for all $0 \leq t_1 \leq \ldots \leq t_k$ and $k \geq 1$, where we set for every $B_k \in \mathcal{B}(E^k)$:

$$\mu_{0,t_1} \times \mu_{t_1,t_2} \times \cdots \times \mu_{t_{k-1},t_k}(x, B_k) :=$$
$$\int\int \cdots \int \mathbf{1}_{B_k}(x_1, \ldots, x_k) \mu_{t_{k-1},t_k}(x_{k-1}, dx_k) \cdots \mu_{t_1,t_2}(x_1, dx_2) \mu_{0,t_1}(x, dx_1).$$

Moreover, for any given Markov transition function μ and initial distribution $\nu \in \mathcal{M}_1(E)$ we can construct an associated Markov process (cf. [Kal97], p.120):

Theorem 3.38 [EXISTENCE] *Let μ be a Markov transition function and $\nu \in \mathcal{M}_1(E)$. Then there exists an E-valued process $X = (X_t : t \geq 0)$ which is a Markov process w.r.t. $(\mathcal{F}_t^X)$, has corresponding Markov transition function μ and satisfies $\mathbb{P} \circ X_0^{-1} = \nu$.*

In the proof of Theorem 3.38 X is constructed as a canonical process, i.e. as the coordinate process $(X_t := \pi_t : t \geq 0)$ of a probability measure, $\mathbb{P}_\nu$, on $[E^{[0,\infty)}, \mathcal{E}^{[0,\infty)}]$. So, given a Markov transition function μ, there exists for every $\nu \in \mathcal{M}_1(E)$ a probability measure $\mathbb{P}_\nu$ on $[E^{[0,\infty)}, \mathcal{E}^{[0,\infty)}]$ under which the coordinate process $X = (X_t : t \geq 0)$ is an $(\mathcal{F}_t^X)_{t \geq 0}$-Markov process with corresponding Markov transition function μ and $\mathbb{P}_\nu[X_0 \in .] = \nu(.)$. More generally, it is not hard to deduce that for every $s \geq 0$ and $\nu \in \mathcal{M}_1(E)$ there is a probability measure $\mathbb{P}_{s,\nu}$ on $[E^{[0,\infty)}, \mathcal{E}^{[s,\infty)}]$ under which the coordinate process $X = (X_t : t \geq s)$ is an $(\mathcal{F}_{[s,t]}^X)_{t \geq s}$-Markov process satisfying $\mu_{r,t}(X_r, B) = \mathbb{P}_{s,\nu}[X_t \in B | X_r]$ $\mathbb{P}_{s,\nu}$-almost

[15]For fixed $0 \leq s \leq t$, the conditional distribution of X_t, given X_s, i.e. $\mathbb{P}[X_t \in .|X_s]$, is said to be *regular* if there exists a probability kernel $\mu_{s,t}$ from E to E which satisfies (3.16); cf. [Kal97] p.84.

surely ($\forall t \geq r \geq s$ and $B \in \mathcal{B}(E)$) and $\mathbb{P}_{s,\nu}[X_s \in .] = \nu(.)$. Note that $\mathcal{E}^{I_0} = \mathcal{F}^X_{I_0}$ for any $I_0 \subset [0,\infty)$ since X_t was defined to be the coordinate projection π_t. The system

$$\Big\{ \Big[E^{[0,\infty)}, \mathcal{E}^{[s,\infty)}, (\mathcal{F}^X_{[s,t]})_{t\geq s}, (X_t : t \geq s), \mathbb{P}_{s,\nu} \Big] : s \geq 0, \nu \in \mathcal{M}_1(E) \Big\}$$

is called *canonical Markov process* (corresponding to μ). For brevity we refer to it as $X = [X, \mathbb{P}_{s,\nu} : s \geq 0, \nu \in \mathcal{M}_1(E)]$. If $(X_t : t \geq s)$ is $\mathbb{P}_{s,\nu}$-almost surely cadlag for every $s \geq 0$ and $\nu \in \mathcal{M}_1(E)$, then we may assume the laws $\mathbb{P}_{s,\nu}$ to be defined on $[D([0,\infty),E), \mathcal{E}^{[s,\infty)} \cap D([0,\infty),E)]$ instead of $[E^{[0,\infty)}, \mathcal{E}^{[s,\infty)}]$ and, accordingly, X_t to be the restriction of π_t to $D([0,\infty),E)$ (the same arguments as used in Section 3.2 do trick). We then call the corresponding system *canonical cadlag Markov process*. Analogously, if $D([0,\infty),E)$ is replaced by $C([0,\infty),E)$, then the system is called *canonical continuous Markov process*. If $\nu = \delta_x$ for some $x \in E$, then we write $\mathbb{P}_{s,x}$ instead of $\mathbb{P}_{s,\delta_x}$. Using this notation we have $\mathbb{P}_{s,x}[X_s = x] = 1$ and

$$\mathbb{P}_{s,x}[X_t \in .] = \mu_{s,t}(x,.) \qquad \forall\, 0 \leq s \leq t,\ x \in E. \tag{3.18}$$

The map $(x,B) \mapsto \mathbb{P}_{s,x}[B]$ is a probability kernel from E to $[E^{[0,\infty)}, \mathcal{E}^{[s,\infty)}]$. In particular, $x \mapsto \mathbb{E}_{s,x}[Y]$ is an element of $B(E)$ for every $[\mathcal{E}^{[s,\infty)}, \mathcal{B}(\mathbb{R})]$-measurable Y. Moreover, $\mathbb{P}_{s,\nu}[.] = \int \mathbb{P}_{s,x}[.]\nu(dx)$ (cf. Lemma 7.7 of [Kal97]). A canonical cadlag Markov process is called *canonical right Markov process* if for every $0 \leq s < t$, $\nu \in \mathcal{M}_1(E)$ and $f \in C_b^+(E)$:

$$[s,t) \ni r \mapsto \int_E f(y)\mu_{r,t}(X_r, dy) = \mathbb{E}_{r,X_r}[f(X_t)] \text{ is } \mathbb{P}_{s,\nu}\text{-a.s. right-continuous.} \tag{3.19}$$

If X is a canonical Markov process, then the coordinate process $(X_t : t \geq s)$ is a Markov process w.r.t. $(\mathcal{F}^X_{[s,t]}) \equiv (\mathcal{F}^X_{[s,t]})_{t\geq s}$ under $\mathbb{P}_{s,\nu}$, for every $s \geq 0$ and $\nu \in \mathcal{M}_1(E)$. Occasionally one wishes the filtration to satisfy the usual conditions. This is not always the case but can be assured by switching to the usual augmentation $(\bar{\mathcal{F}}^{X,\mathbb{P}_{s,\nu}}_{[s,t]})$ of $(\mathcal{F}^X_{[s,t]})$ w.r.t. $\mathbb{P}_{s,\nu}$. Of course, $(X_t : t \geq s)$ might fail to be a Markov process w.r.t. the usual augmentation. However, if X is a right Markov process, this can be excluded:

Proposition 3.39 [CHANGE OF FILTRATION] *Let $X = [X, \mathbb{P}_{s,\nu} : s \geq 0, \nu \in \mathcal{M}_1(E)]$ be an E-valued canonical right Markov process. Then, for every $s \geq 0$ and $\nu \in \mathcal{M}_1(E)$, the coordinate process $(X_t : t \geq s)$ under $\mathbb{P}_{s,\nu}$ is a Markov process not only w.r.t. $(\mathcal{F}^X_{[s,t]})$ but also w.r.t. the $\mathbb{P}_{s,\nu}$-completion $(\tilde{\mathcal{F}}^{X,\mathbb{P}_{s,\nu}}_{[s,t]})$, and also w.r.t. the universal completion $(\tilde{\mathcal{F}}^X_{[s,t]})$, where $\tilde{\mathcal{F}}^X_{[s,t]} := \cap_\nu \tilde{\mathcal{F}}^{X,\mathbb{P}_{s,\nu}}_{[s,t]}$. Moreover, $(\tilde{\mathcal{F}}^{X,\mathbb{P}_{s,\nu}}_{[s,t]})$ coincides with the usual augmentation $(\bar{\mathcal{F}}^{X,\mathbb{P}_{s,\nu}}_{[s,t]})$ w.r.t. $\mathbb{P}_{s,\nu}$, and $(\tilde{\mathcal{F}}^X_{[s,t]})$ coincides with the universal augmentation $\bar{\mathcal{F}}^X_{[s,t]} := \tilde{\mathcal{F}}^X_{[s,t+]}$.*

(One can generalize the proof for the case of Feller-Markov processes which can be found in [RY98] p.93-95; see also [Dyn94] p.27). A canonical Markov process X is called a *strong Markov process* if for every $s \geq 0$ and $\nu \in \mathcal{M}_1(E)$:

$$\mathbb{P}_{s,\nu}[X_{\tau+h} \in B | \mathcal{F}_{[s,\tau]}] = \mathbb{P}_{s,\nu}[X_{\tau+h} \in B | X_\tau] \quad \mathbb{P}_{s,\nu}\text{-almost surely on } \{s \leq \tau < \infty\}$$

for all $h \geq 0$, $B \in \mathcal{B}(E)$ and each $(\mathcal{F}_{[s,t]})_{t\geq s}$-stopping time τ. It is remarkable that any right Markov process is strongly Markovian, cf. [Kuz84].

Remark 3.40 *Any (canonical) right Markov process is a strong Markov process.*

For further analysis we assume E to be a locally compact complete and separable metric space (which is in particular a Polish space). Let $C_0(E)$ denote the space of real-valued continuous functions on E vanishing at infinity, i.e. the space of functions $f \in C(E)$ such that for every $\epsilon > 0$ the set $\{x \in E : |f(x)| \geq \epsilon\}$ is compact. $C_0(E)$ is a Banach space w.r.t. $\|.\|_\infty$. An E-valued Markov process X is said to be *time-homogeneous* if its Markov transition function is stationary, i.e. if $\mu_{0,t} = \mu_{s,s+t}$ for all $s, t \geq 0$. In this case we set $\mu_t := \mu_{0,t}$ for all $t \geq 0$ and $\mathbb{P}_x := \mathbb{P}_{0,x}$, $\mathbb{P}_\nu := \mathbb{P}_{0,\nu}$ for all $x \in E$, $\nu \in \mathcal{M}_1(E)$. Also,

$$P_t f(x) := \int f(y)\mu_t(x, dy), \qquad t \geq 0, x \in E, f \in C_0(E). \tag{3.20}$$

Then $(P_t) \equiv (P_t)_{t\geq 0}$ is a semigroup of linear operators acting on functions. A time-homogeneous canonical cadlag Markov process is called *Feller-Markov process* if the map $x \mapsto \mathbb{E}_x[f(X_t)] = P_t f(x)$ is in $C_0(E)$ for every $t \geq 0$ and $f \in C_0(E)$. Feller-Markov processes correspond to *Feller semigroups*, cf. Proposition 3.42.

Definition 3.41 [Feller semigroup] *A family of linear operators on the Banach space $(C_0(E), \|.\|_\infty)$ is called Feller semigroup if the following assertions hold:*

(i) $P_0 = \mathbb{I}$, $P_{s+t} = P_s P_t f$, $\forall s, t \geq 0, f \in C_0(E)$ *("semigroup")*

(ii) $t \mapsto P_t f$ $\|.\|_\infty$*-continuous,* $\forall f \in C_0(E)$ *("strongly continuous")*

(iii) $\|P_t f\|_\infty \leq \|f\|_\infty$, $\forall t \geq 0$, $f \in C_0(E)$ *("contractive")*

(iv) $f \geq 0 \Rightarrow P_t f \geq 0$, $\forall t \geq 0$, $f \in C_0(E)$ *("non-negative")*

(v) $(\mathbf{1}, \mathbf{1})$ *is in the bp-closure of* $\{(f, P_t f) : f \in \mathrm{dom}(L)\}$, $\forall t \geq 0$ *("conservative").*

Here *bp-closure* means closure w.r.t. *bp*-convergence. Property (iii) implies in particular that the operator P_t is bounded with operator norm $\|P_t\|_\infty \leq 1$ for every $t \geq 0$. Also note that, under (i), condition (ii) holds if and only if $\lim_{t\downarrow 0} \|P_t f - f\|_\infty = 0$ $\forall f \in C_0(E)$.

Proposition 3.42 [Feller semigroup, Feller-Markov process] *If X is an E-valued Feller-Markov process with transition function μ, then (P_t) defined in (3.20) provides a Feller semigroup on $(C_0(E), \|.\|_\infty)$. Conversely, let (P_t) be a Feller semigroup on $(C_0(E), \|.\|_\infty)$. Then there exists an E-valued Feller-Markov process X with transition function μ such that μ and (P_t) correspond as in (3.20).*

For the proof see [Kal97] p.323-325. The *(infinitesimal) generator* L of a strongly continuous semigroup (P_t) of linear bounded operators on $(C_0(E), \|.\|_\infty)$ is given by

$$Lf := \|.\|_\infty\text{-}\lim_{t\downarrow 0} \frac{1}{t}(P_t - \mathbb{I})f, \qquad f \in \mathrm{dom}(L)$$

where $\mathrm{dom}(L)$ consists of those $f \in C_0(E)$ for which the limit exists. L is a linear operator and $\mathrm{dom}(L)$ is a linear subspace of $C_0(E)$. An important relation is (cf. [Wer00] p.337):

$$P_t f - f = \int_0^t L P_s f ds = \int_0^t P_s L f ds \qquad \forall t \geq 0, f \in \mathrm{dom}(L). \tag{3.21}$$

(Note that generators can also be defined for strongly continuous semigroups on more general Banach spaces than $(C_0(E), \|.\|_\infty)$, and (3.21) remains true in that case.) Proposition 3.42 provided a one-to-one correspondence between Feller-Markov processes and Feller-semigroups. So the following theorem (for a similar result see [GZ00], Theorem 1.7) induces a one-to-one correspondence between Feller-Markov processes and generators.

Theorem 3.43 [Hille-Yosida for Feller semigroups] *A linear operator L is the generator of a Feller semigroup (P_t) on $(C_0(E), \|.\|_\infty)$ if and only if these conditions hold:*

(i) $\mathrm{dom}(L)$ *is dense in* $(C_0(E), \|.\|_\infty)$ *and L is closed,*

(ii) $(\lambda\mathbb{I}-L)^{-1}$ *exists on $C_0(E)$, is positive and bounded with* $\|(\lambda\mathbb{I}-L)^{-1}\|_\infty \leq \frac{1}{\lambda}$, $\forall\lambda > 0$,

(iii) $(\mathbf{1}, \mathbf{0})$ *is in the bp-closure of* $\{(f, Lf) : f \in \mathrm{dom}(L)\}$.

3.10 Additive functionals of Markov processes

This section is devoted to the notion of *continuous additive functionals* (CAF) of Markov processes in the sense of [Dyn94]. Let E be a Polish space and consider some canonical continuous E-valued Markov process $X = [X, \mathbb{P}_{s,\nu} : s \geq 0, \nu \in \mathcal{M}_1(E)]$ (hence $\Omega = C([0,\infty), E)$ and $X_t = \bar{\pi}_t$). Suppose X satisfies (3.19); then X is in particular a right Markov process and even a Hunt process. For any $\mathbb{R}_+$-valued non-decreasing function $[0,\infty) \ni t \mapsto A_t$, we set $A(B) := \int_B dA_r$ for every $B \in \mathcal{B}([0,\infty))$.

Definition 3.44 [continuous additive functional] *A functional*

$$A : [0,\infty) \times C([0,\infty), E) \to \mathbb{R}_+, \qquad (t, f) \mapsto A_t(f)$$

is said to be a continuous additive functional (CAF) of X if the following assertions hold:

(i) $A_0(f) = 0$ *for every* $f \in C([0,\infty), E)$,

(ii) $(A_t(f) : t \geq 0)$ *is non-decreasing for every* $f \in C([0,\infty), E)$,

(iii) $(A((s,t], .) : t > s)$ *is* $\mathbb{P}_{s,\nu}$*-a.s. continuous for every* $s \geq 0$ *and* $\nu \in \mathcal{M}_1(E)$,

(iv) $f \mapsto A((s,t], f)$ *is* $[\tilde{\mathcal{F}}^X_{(s,t]}, \mathcal{B}(\mathbb{R}_+)]$*-measurable for every* $0 \leq s < t$.

Here $\tilde{\mathcal{F}}^X_{(s,t]}$ denotes the completion of $\mathcal{F}^X_{(s,t]}$ w.r.t. $\mathbb{P}_{s,\nu}$ for every $\nu \in \mathcal{M}_1(E)$. A basic example for a CAF is given by

$$A^{g,\varrho}_t(f) = \int_0^t g(r, f(r))\varrho(dr), \qquad t \geq 0 \tag{3.22}$$

for every non-negative continuous function $g : [0,\infty) \times E \to \mathbb{R}$ and every non-atomic $\varrho(dr) \in \mathcal{M}([0,\infty))$. Intuitively a CAF A represents a clock attached to X whose reading on the interval $(s,t]$ depends only on the realization of $(X_r : s < r \leq t)$. CAFs will play an important role in the context of branching diffusions, cf. Section 9.3 below. We stress the fact that the notion of CAFs in the sense of Definition 3.44 differs from the notion of CAF in the sense of [Vol60], [BG68]. In the latter two references CAFs are only considered for time-homogeneous Markov processes X and defined to be functionals $A : [0,\infty) \times C([0,\infty), E) \to \mathbb{R}_+$ satisfying $(i) - (iv)$ of Definition 3.44 (only for $s = 0$) and

$$A_{s+t} = A_s + A_t \circ \theta_s \qquad \mathbb{P}_\nu\text{-almost surely}, \quad \forall s,t \geq 0,\ \nu \in \mathcal{M}_1(E) \tag{3.23}$$

where θ_t is the usual shift operator on the canonical path space. A weakness of this notion is that $A^{g,\varrho}$ defined in (3.22) would be a CAF only for $g(r,x) \equiv g(x)$ and $\varrho(dr) = dr$. But in the sequel we will deal with functionals $A^{g,\varrho}$ where $\varrho(dr)$ differs from dr. Hence, from now on, CAF will always be understood in the sense of Definition 3.44. The *characteristic* h of a CAF A of our canonical Markov process X is defined by

$$h_{s,t}(x) := \begin{cases} \mathbb{E}_{s,x}[A(s,t]] & , \quad 0 \leq s \leq t \\ 0 & , \quad s > t \end{cases}, \qquad x \in E.$$

It is the counterpart to the α-potentials from the theory of CAF in the sense of (3.23). In fact, the characteristic determines the CAF (cf. [Dyn94] Theorem 2.4.1):

Proposition 3.45 [UNIQUENESS] *If A and A' are two CAF of X with the same finite characteristic, then $(A(s,t] : t \geq s)$ and $(A'(s,t] : t \geq s)$ are $\mathbb{P}_{s,\nu}$-indistinguishable for every $s \geq 0$ and $\nu \in \mathcal{M}_1(E)$.*

If μ is the corresponding Markov transition function of X, set $P_{s,t}f(x) := \int f(y)\mu_{s,t}(x,dy)$ for all $0 \leq s \leq t$, $x \in E$ and $f \in B(E)$. $(P_{s,t}) \equiv (P_{s,t})_{0 \leq s \leq t}$ provides an (inhomogeneous) semigroup on $B(E)$. Let $\mathcal{T}_s$ denote the class of all s-stopping times τ (i.e., for every $\nu \in \mathcal{M}_1(E)$, τ has to be a stopping time w.r.t. the usual augmentation $(\bar{\mathcal{F}}_{[s,t]}^{X,\mathbb{P}_{s,\nu}})$ of $(\mathcal{F}_{[s,t]}^X)$). Proposition 3.45 shows that CAFs can be identified with help of their characteristics. The following result concerns the existence of a CAF for a given characteristic.

Proposition 3.46 [EXISTENCE] *Assume $h_{s,t} : E \to \mathbb{R}_+$ is a measurable function for every $0 \leq s \leq t$. Then there exists a CAF A of X with characteristic h if the following assertions hold:*

(i) $h_{s,v} + P_{s,v}h_{v,t} = h_{s,t} \qquad \forall\, 0 \leq s \leq v \leq t,$

(ii) $h_{s,t}(x) \downarrow 0$ *as* $t \downarrow s \qquad \forall\, s \geq 0,\ x \in E,$

(iii) $(h_{\tau,t}(X_\tau) : \tau \in \mathcal{T}_s)$ *uniformly integrable w.r.t.* $\mathbb{P}_{s,x} \qquad \forall\, 0 \leq s \leq t,\ x \in E,$

(iv) $\mathbb{E}_{s,x}[h_{\tau_n,t}(X_{\tau_n})] \to \mathbb{E}_{s,x}[h_{\tau,t}(X_\tau)] \qquad \forall\, \tau_n, \tau \in \mathcal{T}_s : \tau_n \uparrow \tau \qquad \forall\, 0 \leq s \leq t,\ x \in E.$

The proof can be found in [Dyn94] (Theorems 2.4.2). A crucial point in the context of CAFs and their characteristics is the following martingale property:

Lemma 3.47 *Let A be a CAF of X and h be its characteristic. Fix $0 \leq s < T$ and define*

$$M_t^{(s)} := h_{t,T}(X_t) + A(s,t], \qquad t \in [s,T].$$

Then $M^{(s)} = (M_t^{(s)} : t \in [s,T])$ is an $(\tilde{\mathcal{F}}_{[s,t]}^X)$-martingale under $\mathbb{P}_{s,\nu}$ for every $\nu \in \mathcal{M}_1(E)$.

Proof With help of the Markov property of X we obtain for every $t, t+\epsilon \in [s,T]$ ($\epsilon \geq 0$):

$$\begin{aligned}
&\mathbb{E}_{s,\nu}\left[M_{t+\epsilon}^{(s)} - M_t^{(s)}\middle|\tilde{\mathcal{F}}_{[s,t]}^X\right] \\
&= \mathbb{E}_{s,\nu}\left[\mathbb{E}_{t+\epsilon,B_{t+\epsilon}}\left[A(t+\epsilon,T)\right] - \mathbb{E}_{t,B_t}\left[A(t,T)\right] + A(s,t+\epsilon] - A(s,t]\middle|\tilde{\mathcal{F}}_{[s,t]}^X\right] \\
&= \mathbb{E}_{s,\nu}\left[\mathbb{E}_{t+\epsilon,B_{t+\epsilon}}\left[A(t+\epsilon,T]\right]\middle|\tilde{\mathcal{F}}_{[s,t]}^X\right] - \mathbb{E}_{t,B_t}\left[A(t,T]\right] + \mathbb{E}_{s,\nu}\left[A(t,t+\epsilon]\middle|\tilde{\mathcal{F}}_{[s,t]}^X\right] \\
&= \mathbb{E}_{t,B_t}\left[\mathbb{E}_{t+\epsilon,B_{t+\epsilon}}\left[A(t+\epsilon,T]\right]\right] - \mathbb{E}_{t,B_t}\left[A(t,T]\right] + \mathbb{E}_{t,B_t}\left[A(t,t+\epsilon]\right] \\
&= \mathbb{E}_{t,B_t}\left[A(t+\epsilon,T]\right] - \mathbb{E}_{t,B_t}\left[A(t+\epsilon,T]\right] \;=\; 0 \qquad \mathbb{P}_{s,\nu}\text{-almost surely.}
\end{aligned}$$

This proves the claim. □

3.11 Measure-valued Markov processes

In the setting of Section 3.9 assume the state space E to be the space $\mathcal{M}_f(S)$ of finite Borel measures on some complete and separable metric space S. From Lemma 2.12 we know that $\mathcal{M}_f(S)$ equipped with the weak topology is Polish. We had seen in Section 3.9 that the law of a Markov process is determined by the corresponding Markov transition function μ and the initial distribution. On the other hand, by Proposition 3.31 the Markov transition function $\mu = \{\mu_{s,t} : 0 \leq s \leq t\}$ on $\mathcal{M}_f(S)$ is determined by the Laplace transforms $L_{\mu_{s,t}(\eta,d\nu)}$ ($0 \leq s \leq t$, $\eta \in \mathcal{M}_f(S)$) of the transition probabilities $\mu_{s,t}(\eta, d\nu)$. Hence,

Remark 3.48 [IDENTIFICATION] *The law of an $\mathcal{M}_f(S)$-valued Markov process X is determined by the Laplace transforms $L_{\mu_{s,t}(\eta,d\nu)}$ ($0 \leq s \leq t$, $\eta \in \mathcal{M}_f(S)$) and the initial distribution $\mathbb{P} \circ X_0^{-1}$, where $\mu = \{\mu_{s,t} : 0 \leq s \leq t\}$ denotes the corresponding Markov transition function of X.*

Our intension now is to construct an $\mathcal{M}_f(S)$-valued Markov process for given Laplace transforms of the transition probabilities and a given initial distribution. The key tool will be Theorem 3.38. Before turning to the construction we briefly focus on positive and negative definite functionals. Recall that a functional $L : C_b^+(S) \to \mathbb{R}$ is said to be positive definite if $\sum_{i,j=1}^k \lambda_i\lambda_j L(\psi_i + \psi_j) \geq 0$ for all $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$, $\psi_1, \ldots, \psi_k \in C_b^+(S)$ and $k \geq 1$. We say L is *negative definite* if $\sum_{i,j=1}^k \lambda_i\lambda_j L(\psi_i + \psi_j) \leq 0$ whenever $k \geq 2$ and the $\lambda_1, \ldots, \lambda_k$ sum to zero. The following lemma is taken from [BCR84] (p.74).

Lemma 3.49 *A functional $L : C_b^+(S) \to \mathbb{R}$ is negative definite if and only if $e^{-\theta L}$ is positive definite for all $\theta > 0$.*

The Lemma is known as *Schoenberg's theorem*. Let us now consider an (inhomogeneous) *bp*-continuous semigroup $(U_{s,t})$ of negative definite operators on $C_b^+(S)$, i.e.

(0) $U_{s,t} : C_b^+(S) \to C_b^+(S), \quad \forall 0 \leq s \leq t$

(i) $U_{s,s} = \mathbb{I}, \quad U_{s,t} = U_{s,v}U_{v,t}, \quad \forall 0 \leq s \leq v \leq t$

(ii) $\psi \mapsto U_{s,t}\psi$ is *bp*-continuous, $\quad \forall 0 \leq s \leq t$

(iii) $\psi \mapsto U_{s,t}\psi(x)$ is negative definite, $\quad \forall 0 \leq s \leq t$ and $x \in S$

and define $L^{\eta}_{s,t}(\psi) := \exp(-\langle \eta, U_{s,t}\psi \rangle)$ for all $0 \leq s \leq t$, $\psi \in C_b^+(S)$ and $\eta \in \mathcal{M}_f(S)$. The following result ensures the existence of an inhomogeneous $\mathcal{M}_f(S)$-valued Markov process whose transition probability $\mu_{s,t}(\eta, d\nu)$ has Laplace transform $L_{\mu_{s,t}(\eta,d\nu)}(\psi) = L^{\eta}_{s,t}(\psi)$. This Markov process is in particular a so-called strong branching process, cf. Section 9.1.

Proposition 3.50 [EXISTENCE] *Let $(U_{s,t})$ and $(L^{\eta}_{s,t})$ be as above and $\Upsilon \in \mathcal{M}_1(\mathcal{M}_f(S))$. Then there exists a (time-inhomogeneous) $\mathcal{M}_f(S)$-valued Markov process X whose initial distribution $\mathbb{P} \circ X_0{}^{-1}$ is Υ and whose corresponding Markov transition function μ is determined by $L_{\mu_{s,t}(\eta,d\nu)}(\psi) = L^{\eta}_{s,t}(\psi)$ for $0 \leq s \leq t$, $\psi \in C_b^+(S)$, $\eta \in \mathcal{M}_f(S)$.*

Proof $L^{\eta}_{s,t}$ is clearly a functional from $C_b^+(S)$ to $[0,1]$. Also, it satisfies the assumptions of Proposition 3.34. The positive definiteness is not obvious but a consequence of Lemma 3.49 and the fact that $\psi \mapsto \langle \eta, U_{s,t}\psi \rangle$ is negative definite (which easily follows from property *(iii)*). Then, by Proposition 3.34 there exists a probability measure $\mu_{s,t}(\eta, d\nu)$ on $\mathcal{M}_f(S)$ with Laplace transform $L_{\mu_{s,t}(\eta,d\nu)} = L^{\eta}_{s,t}$, for every $0 \leq s \leq t$ and $\eta \in \mathcal{M}_f(S)$. If we could show that $\mu = \{(\mu_{s,t}(\eta, d\nu))_{\eta \in \mathcal{M}_f(S)} : 0 \leq s \leq t\}$ provides a Markov transition function, then the claim followed by Theorem 3.38.

In the remainder of the proof we establish that μ is a Markov transition function. First we show, for every $0 \leq s \leq t$, that $\mu_{s,t}$ is a probability kernel from $\mathcal{M}_f(S)$ to $\mathcal{M}_f(S)$ or, equivalently, that $\eta \mapsto \mu_{s,t}(\eta, .)$ is $[\mathcal{B}(\mathcal{M}_f(S)), \mathcal{B}(\mathcal{M}_1(\mathcal{M}_f(S)))]$-measurable. In order to prove the latter measurability, it suffices to show sequential continuity w.r.t. the corresponding weak topologies[16]. We shall apply Theorem 3.2.6 of [Daw93] which says that $\{\Psi_\psi : \psi \in C_b^+(S)\} \subset C_b(\mathcal{M}_f(S))$ is weak convergence determining in $\mathcal{M}_1(\mathcal{M}_f(S))$, where $\Psi_\psi(\nu) := e^{-\langle \nu, \psi \rangle}$. That is, $(\Theta_n) \subset \mathcal{M}_1(\mathcal{M}_f(S))$ converges weakly to $\Theta \in \mathcal{M}_1(\mathcal{M}_f(S))$ if and only if

$$\langle \Theta_n, \Psi_\psi \rangle = L_{\Theta_n}(\psi) \to L_{\Theta_n}(\psi) = \langle \Theta, \Psi_\psi \rangle \qquad \forall \psi \in C_b^+(S).$$

Sequential weak continuity of the map $\eta \mapsto \mu_{s,t}(\eta, .)$ means that for every $\eta \in \mathcal{M}_f(S)$ and $(\eta_n) \subset \mathcal{M}_f(S)$ converging weakly to η we have weak convergence of $\mu_{s,t}(\eta_n, .)$ to $\mu_{s,t}(\eta, .)$. Now, pick $\eta \in \mathcal{M}_f(S)$ and let (η_n) be any sequence in $\mathcal{M}_f(S)$ converging weakly to η. Thus $\langle \eta_n, \phi \rangle \to \langle \eta, \phi \rangle$ and so $e^{-\langle \eta_n, \phi \rangle} \to e^{-\langle \eta, \phi \rangle}$ for every $\phi \in C_b^+(S)$. In particular we obtain $L^{\eta_n}_{s,t}(\psi) \to L^{\eta}_{s,t}(\psi)$ and so

$$\langle \mu_{s,t}(\eta_n, d\nu), \Psi_\psi \rangle = L_{\mu_{s,t}(\eta_n,d\nu)}(\psi) \to L_{\mu_{s,t}(\eta,d\nu)}(\psi) = \langle \mu_{s,t}(\eta, d\nu), \Psi_\psi \rangle \quad \forall \psi \in C_b^+(S).$$

That is, $\mu_{s,t}(\eta_n, d\nu)$ converges weakly to $\mu_{s,t}(\eta, d\nu)$. Consequently, the map $\eta \mapsto \mu_{s,t}(\eta, .)$ is weakly continuous (at each η), i.e. $\mu_{s,t}$ is a probability kernel from $\mathcal{M}_f(S)$ to $\mathcal{M}_f(S)$.

[16] In Section 2.6 we have seen that there exist metrics on $\mathcal{M}_f(S)$ and $\mathcal{M}_1(\mathcal{M}_f(S))$ which induce the corresponding weak topologies. This implies that (topological) continuity is equivalent to (topological) sequential continuity.

It remains to show (i) and (ii) of Definition 3.36. Assertion (i) is an immediate consequence of $U_{s,s} = \mathbb{I}$. To verify (ii) one can exploit the semigroup property of $(U_{s,t})$. In fact, for every $0 \le s \le v \le t$ we have

$$
\begin{aligned}
L_{\mu_{s,t}(\eta,d\nu)}(\psi) &= e^{-\langle \eta, U_{s,t}\psi\rangle} = e^{-\langle \eta, U_{s,v}U_{v,t}\psi\rangle} = L_{\mu_{s,v}(\eta,d\lambda)}(U_{v,t}\psi) \\
&= \int e^{-\langle \lambda, U_{v,t}\psi\rangle}\mu_{s,v}(\eta,d\lambda) = \int L_{\mu_{v,t}(\lambda,d\nu)}(\psi)\mu_{s,v}(\eta,d\lambda) \\
&= \int\int e^{-\langle \nu,\psi\rangle}\mu_{v,t}(\lambda,d\nu)\mu_{s,v}(\eta,d\lambda) = L_{\mu_{v,t}(\lambda,d\nu)\mu_{s,v}(\eta,d\lambda)}(\psi)
\end{aligned}
$$

and so Proposition 3.31 implies the desired Chapman-Kolmogorov equation. $\square$

Proposition 3.50 provides in particular a canonical $\mathcal{M}_f(S)$-valued Markov process $X = [X, \mathbb{P}_{s,\Upsilon} : s \ge 0, \Upsilon \in \mathcal{M}_1(\mathcal{M}_f(S))]$ satisfying $\mathbb{E}_{s,\eta}[e^{-\langle X_t,\psi\rangle}] = \exp(-\langle \eta, U_{s,t}\psi\rangle)$ for all $0 \le s \le t$, $\psi \in C_b^+(S)$ and $\eta \in \mathcal{M}_f(S)$. The next result shows that, if X is known to be cadlag and $[0,t) \ni s \mapsto U_{s,t}\psi$ to be right-continuous for every t and ψ, then X is a right Markov process and so, by Remark 3.40, also a strong Markov process.

Proposition 3.51 [RIGHT AND STRONG MARKOV PROPERTY] *Suppose $[0,t) \ni s \mapsto U_{s,t}\psi$ is right-continuous w.r.t. $\|.\|_\infty$ for every $t > 0$ and $\psi \in C_b^+(S)$. Assume that the canonical $\mathcal{M}_f(S)$-valued Markov process $X = [X, \mathbb{P}_{s,\Upsilon} : s \ge 0, \Upsilon \in \mathcal{M}_1(\mathcal{M}_f(S))]$, induced by Proposition 3.50, is cadlag. Then X is a right (and in particular a strong) Markov process.*

Proof We have to show (3.19). We shall even show that the right-continuity in (3.19) holds for *all* samples. In fact, the calculations below do not depend on the particular law $\mathbb{P}_{s,\Upsilon}$ on $D([0,\infty), \mathcal{M}_f(S))$. We only exploit the fact that *all* samples of $r \mapsto \bar{X}_r(dx)$ are cadlag. As in the previous proof, we set $\Psi_\psi(\nu) := e^{-\langle \nu,\psi\rangle}$ for $\psi \in C_b^+(S)$ and note that $\Psi_\psi \in C_b(\mathcal{M}_f(S))$. Pick $r \in [0,t)$ and $(r_n) \subset [r,t)$ with $r_n \downarrow r$. Then we have for every $\psi \in C_b^+(S)$:

$$
\begin{aligned}
&\Big|\int \Psi_\psi(\nu)\mu_{r,t}(X_r,d\nu) - \int \Psi_\psi(\nu)\mu_{r_n,t}(X_{r_n},d\nu)\Big| \\
&= \Big|L_{\mu_{r,t}(X_r,d\nu)}(\psi) - L_{\mu_{r_n,t}(X_{r_n},d\nu)}(\psi)\Big| \;\Big(= \Big|\mathbb{E}_{r,X_r}\Big[e^{-\langle X_t,\psi\rangle}\Big] - \mathbb{E}_{r_n,X_{r_n}}\Big[e^{-\langle X_t,\psi\rangle}\Big]\Big|\;\Big) \\
&= \Big|L_{r,t}^{X_r}(\psi) - L_{r_n,t}^{X_{r_n}}(\psi)\Big| = \Big|e^{-\langle X_r, U_{r,t}\psi(.)\rangle} - e^{-\langle X_{r_n}, U_{r_n,t}\psi(.)\rangle}\Big| \\
&\le |\langle X_r, U_{r,t}\psi(.)\rangle - \langle X_{r_n}, U_{r,t}\psi(.)\rangle| + |\langle X_{r_n}, U_{r,t}\psi(.)\rangle - \langle X_{r_n}, U_{r_n,t}\psi(.)\rangle| \\
&\le |\langle X_r, U_{r,t}\psi(.)\rangle - \langle X_{r_n}, U_{r,t}\psi(.)\rangle| + \sup_{r' \in [0,t)} \langle X_{r'}, \mathbf{1}\rangle \, \|U_{r,t}\psi(.) - U_{r_n,t}\psi(.)\|_\infty \to 0
\end{aligned}
$$

(as $n \to \infty$). Here the first summand converges to 0 as $n \to \infty$ by the right-continuity of X w.r.t. the weak topology. The second summand converges to 0 as $n \to \infty$ by the right-continuity of $r \mapsto U_{r,t}\psi$ w.r.t. $\|.\|_\infty$ and by $\sup_{r'\in[0,t)}\langle X_{r'}, \mathbf{1}\rangle < \infty$ (which holds since $r' \mapsto \langle X_{r'}, \mathbf{1}\rangle$ is cadlag). Hence, the map $[0,t) \ni r \mapsto \int \Psi_\psi(\nu)\mu_{r,t}(X_r,d\nu)$ is right-continuous for all $\psi \in C_b^+(S)$. From [Daw93] (Theorem 3.2.6) we know that $\{\Psi_\psi : \psi \in C_b^+(S)\}$ is weak convergence determining in $\mathcal{M}_1(\mathcal{M}_f(S))$. So, since right-continuity is equivalent to sequential right-continuity in metric spaces, we can easily deduce right-continuity of $[0,t) \ni r \mapsto \int f(\nu)\mu_{r,t}(X_r,d\nu)$ for all $f \in C_b(\mathcal{M}_f(S))$. In particular, (3.19) holds. $\square$

4 Auxiliary lemmas

This chapter provides a number of auxiliary lemmas which will be needed later on. The first section concerns estimates for certain measure potentials and related expressions. Note that in the field of measure theory $U_\alpha(x) := \int_{\mathbb{R}^n} |x-y|^{-\alpha}\mu(dy)$ is sometimes called *α-potential* of $\mu(dy)$ at x. In the second section we focus on (integrals of the) increments of the heat kernel and the third section is devoted to some Growall-type lemmas.

4.1 Measure potential estimates

A basic result in integration theory is the following lemma (cf., for instance, [Mat95] p.15). It will be the main tool for the proofs of Lemmas 4.2 and 4.4.

Lemma 4.1 *Let $n \geq 1$, $\nu(dx) \in \mathcal{M}(\mathbb{R}^n)$ and $g : \mathbb{R}^n \to \mathbb{R}_+$ Borel measurable. Then,*

$$\int_{\mathbb{R}^n} g(x)\nu(dx) = \int_0^\infty \nu(x \in \mathbb{R}^n : g(x) \geq u)du.$$

Lemma 4.2 *Let $d \geq 1$, $\mu_1(dx) \in \mathcal{M}_{uni}(\mathbb{R}^d)$, $\lambda > 0$ and $\alpha_1, \alpha_1' \in (0,d]$. Consider the following five statements:*

(i) $\exists c > 0 :\ \sup_{x\in\mathbb{R}^d} \mu_1(B[x,r]) \leq c\, r^{\alpha_1} \quad \forall r \in (0,1]$

(ii) $\exists c > 0 :\ \sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{r}} \mu_1(dy) \leq c\, r^{\alpha_1/2} \quad \forall r \in (0,1]$

(iii) $\exists c > 0 :\ \sup_{x\in\mathbb{R}^d} \int_{B[x,1]} |x-y|^{-\alpha_1'}\mu_1(dy) < \infty$

(iv) $\exists c_\lambda > 0 :\ \sup_{x\in\mathbb{R}^d}\ e^{\mp\lambda|x|} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{r}} e^{\pm\lambda|y|}\mu_1(dy) \ \leq\ c_\lambda\, r^{\alpha_1/2} \quad \forall r \in (0,1]$

(v) $\exists c_\lambda > 0 :\ \sup_{x,x'\in\mathbb{R}^d}\ e^{-\lambda|x-x'|}e^{-\lambda|x|} \int_{\mathbb{R}^d} e^{-\frac{|x'-y|^2}{r}} e^{\lambda|y|}\mu_1(dy) \leq c_\lambda\, r^{\alpha_1/2} \ \forall r \in (0,1]$.

Then, (iii) $\Rightarrow$ (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iv), (v) with $\alpha_1 = \alpha_1'$. Moreover, (i) $\Rightarrow$ (iii) when $\alpha_1 > \alpha_1'$.

Proof (ii)$\Rightarrow$(i): Assuming (ii) we obtain for all $r \in (0,1]$:

$$e^{-1}\mu_1(B[x,\sqrt{r}]) = \int_{\mathbb{R}^d} e^{-1}\mathbf{1}_{B[x,\sqrt{r}]}(y)\mu_1(dy) \leq \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{r}}\mu_1(dy) \leq c\, r^{\alpha_1/2}.$$

(i)$\Rightarrow$(ii): Using Lemma 4.1 we obtain for $r \in (0,1]$:

$$\begin{aligned}\int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{r}}\mu_1(dy) &= \int_0^\infty \mu_1(y : e^{-\frac{|x-y|^2}{r}} \geq u)du = \int_0^1 \mu_1(B[x,(r\log\frac{1}{u})^{1/2}])du \\ &\leq \int_0^1 c\Big((r\log\frac{1}{u})^{1/2}\Big)^{\alpha_1} du = c\int_0^1 (\log\frac{1}{u})^{\alpha_1/2}du\, r^{\alpha_1/2} = \tilde{c}\, r^{\alpha_1/2}.\end{aligned}$$

(iii)$\Rightarrow$(i): One easily estimates for $r \in (0,1]$:

$$c \geq \int_{B[x,1]} |x-y|^{-\alpha_1}\mu_1(dy) \geq \int_{B[x,r]} r^{-\alpha_1}\mu_1(dy) = r^{-\alpha_1}\mu_1(B[x,r]).$$

(i)⇒(iii): By Lemma 4.1,

$$\begin{aligned}
\int_{B[x,1]} |x-y|^{-\alpha_1'} \mu_1(dy) &= \int_0^\infty \mu_1(y : \mathbf{1}_{B[x,1]}(y)|x-y|^{-\alpha_1'} \geq u)du \\
= \int_0^\infty \mu_1(y : \mathbf{1}_{B[x,1]}(y) u^{-1/\alpha_1'} \geq |x-y|)du &= \int_0^\infty \mu_1(B[x, u^{-1/\alpha_1'} \wedge 1])du \\
\leq \int_0^1 \mu_1(B[x,1])du + \int_1^\infty c\,(u^{-1/\alpha_1'})^{\alpha_1} du &= \mu_1(B[x,1]) + c \int_1^\infty u^{-\alpha_1/\alpha_1'} du
\end{aligned}$$

which is finite whenever $\alpha_1 > \alpha_1'$.

(ii)⇒(iv): We obtain for $r \in (0,1]$:

$$\begin{aligned}
&\int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{r}} e^{\pm\lambda|y|} \mu_1(dy) = \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{r}} e^{\lambda(\pm|y| \mp |x|)} \mu_1(dy) e^{\pm\lambda|x|} \\
&\leq \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{r}} e^{\lambda|x-y|} \mu_1(dy) e^{\pm\lambda|x|} \\
&= \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{2r}} e^{-\frac{|x-y|^2 - 2r\lambda|x-y|}{2r}} \mu_1(dy) e^{\pm\lambda|x|} \\
&\leq \int_{B[x,2r\lambda]^c} e^{-\frac{|x-y|^2}{2r}} \mu_1(dy) e^{\lambda|x|} + \int_{B[x,2r\lambda]} e^{-\frac{|x-y|^2}{2r}} e^{\lambda 2r\lambda} \mu_1(dy) e^{\pm\lambda|x|} \\
&\leq c\,(2r)^{\alpha_1/2}\, e^{\pm\lambda|x|} + c\, e^{2\lambda^2}\, (2r)^{\alpha_1/2}\, e^{\pm\lambda|x|} \;\leq\; c_\lambda\, r^{\alpha_1/2}\, e^{\pm\lambda|x|}.
\end{aligned}$$

(iv)⇒(v) is trivial since $e^{\lambda|x'|} \leq e^{\lambda|x|} e^{\lambda|x-x'|}$. □

Remark 4.3 *If we assume (i) of Lemma 4.2, then we also obtain*

(ii)' $\quad \exists c > 0 : \quad \sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{r}} \mu(dy) \leq c\,(r^{d/2} \vee r^{\alpha_1/2}) \qquad \forall r > 0.$

This is true since elements of $\mathcal{M}_{uni}(\mathbb{R}^d)$ are globally bounded (i.e. on balls with radius bigger than 1) by a multiple of the Lebesgue measure (mimic the proof of Lemma 4.2(i) ⇒ (ii)). In particular, (i) of Lemma 4.2 implies (ii) of Lemma 4.2 for all $r \in (0,T]$ when the constant c is replaced by a suitable constant c_T depending on T.

Lemma 4.4 *Let $\mu_2(dt) \in \mathcal{M}([0,\infty))$. Assume there exists an $\alpha_2 \in (0,1]$ such that*

$$\forall T > 0\ \exists c_T > 0 : \quad \sup_{t\leq T} \mu_2([0,\infty) \cap B[t,r]) \leq c_T\, r^{\alpha_2} \quad \forall r \in (0,1].$$

Then, for every $T > 0$ there are finite constants c_T', c_T'', c_T''' and $c_T'''' > 0$ such that the following inequalities hold for all $0 \leq t \leq T$:

(i) $\int_s^t \frac{1}{(t-r)^\gamma} \mu_2(dr) \leq c_T'\,(t-s)^{\alpha_2-\gamma} \qquad \forall \gamma \in [0,\alpha_2], 0 \leq s \leq t$

(ii) $\int_s^v \frac{1}{(t-r)^\gamma} \mu_2(dr) \leq c_T''\,(t-v)^{-(\gamma-\alpha_2)} \qquad \forall \gamma \in (\alpha_2,\infty), 0 \leq s \leq v < t$

(iii) $\int_0^t \frac{r^\delta}{(t-r)^\gamma} \mu_2(dr) \leq c_T'''\, t^{\delta+\alpha_2-\gamma}(\theta^\delta + (1-\theta)^{\alpha_2-\gamma}) \qquad \forall \gamma \in [0,\alpha_2], \theta \in [0,1], \delta > 0$

(iv) $\int_0^T e^{-\gamma r}\mu_2(dr) \le c''''_T\, \gamma^{-\alpha_2} \qquad \forall\gamma > 0.$

Proof (i) By means of Lemma 4.1 and a substitution $v = u^{-1/\gamma}$ we obtain

$$\begin{aligned}
\int_s^t \frac{1}{(t-r)^\gamma}\mu_2(dr) &= \int_0^\infty \mu_2\Big(r : \mathbf{1}_{[s,t]}(r)\frac{1}{(t-r)^\gamma} \ge u\Big)du \\
&\le \int_0^{(t-s)^{-\gamma}} \mu_2([s,t])du + \int_{(t-s)^{-\gamma}}^\infty \mu_2\Big(r : \mathbf{1}_{[s,t]}(r)\frac{1}{(t-r)^\gamma} \ge u\Big)du \\
&\le (t-s)^{-\gamma}\, c_T(t-s)^{\alpha_2} + \int_{(t-s)^{-\gamma}}^\infty \mu_2([t-u^{-1/\gamma}, t])du \\
&= c_T(t-s)^{\alpha_2-\gamma} + \int_0^{t-s} \gamma v^{-\gamma-1}\mu_2\Big([t-v,t]\Big)dv \\
&\le c_T(t-s)^{\alpha_2-\gamma} + \int_0^{t-s} \gamma v^{-\gamma-1}\, c_T v^{\alpha_2} dv \ \le c'_T\, (t-s)^{\alpha_2-\gamma}.
\end{aligned}$$

(ii) The proof goes along the lines of the proof of (i) with the obvious changes.
(iii) Elementary estimates and part (i) yield

$$\begin{aligned}
\int_0^t \frac{r^\delta}{(t-r)^\gamma}\mu_2(dr) &= \int_0^{\theta t} \frac{r^\delta}{(t-r)^\gamma}\mu_2(dr) + \int_{\theta t}^t \frac{r^\delta}{(t-r)^\gamma}\mu_2(dr) \\
&\le (\theta t)^\delta \int_0^{\theta t} \frac{1}{(t-r)^\gamma}\mu_2(dr) + t^\delta \int_{\theta t}^t \frac{1}{(t-r)^\gamma}\mu_2(dr) \\
&\le (\theta t)^\delta \int_0^{t} \frac{1}{(t-r)^\gamma}\mu_2(dr) + t^\delta \int_{\theta t}^t \frac{1}{(t-r)^\gamma}\mu_2(dr) \\
&\le (\theta t)^\delta c_T t^{\alpha_2-\gamma} + t^\delta c_T((1-\theta)t)^{\alpha_2-\gamma} \ \le\ c_T t^{\delta+\alpha_2-\gamma}(\theta^\delta + (1-\theta)^{\alpha_2-\gamma}).
\end{aligned}$$

(iv) With help of Lemma 4.1 and a substitution $v = \log\frac{1}{u}$ we obtain

$$\int_0^T e^{-\gamma r}\mu_2(dr) = \int_0^\infty \mu_2\Big(r : \mathbf{1}_{[0,T]}(r)e^{-\gamma r} \ge u\Big)du \le \int_0^1 \mu_2\Big([0, T\wedge\frac{1}{\gamma}\log\frac{1}{u}]\Big)du =$$

$$\int_0^\infty \mu_2\Big([0,T\wedge\frac{1}{\gamma}v]\Big)e^{-v}dv = \int_0^{T\gamma} \mu_2\Big([0,T\wedge\frac{1}{\gamma}v]\Big)e^{-v}dv + \int_{T\gamma}^\infty \mu_2\Big([0,T\wedge\frac{1}{\gamma}v]\Big)e^{-v}dv$$

$$\le \int_0^{T\gamma} c_T\Big(\frac{1}{\gamma}v\Big)^{\alpha_2} e^{-v}dv + c_T\, T^{\alpha_2}\int_{T\gamma}^\infty e^{-v}dv \ \le\ c_T\,\gamma^{-\alpha_2} + c_T\, T^{\alpha_2}e^{-T\gamma} \ \le\ c_{T,\alpha_2}\gamma^{-\alpha_2}$$

where the last inequality follows from $e^{-T\gamma} \le c_{T,\alpha_2}\,\gamma^{-\alpha_2}\ \forall\gamma > 0.$ □

4.2 Heat kernel estimates

Let p denote the *heat kernel* in $\mathbb{R}^d$ which is defined by

$$p_t(x,y) := (2\pi t)^{-d/2}e^{-\frac{|x-y|^2}{2t}} \qquad \forall t > 0,\ x,y \in \mathbb{R}^d.$$

Lemma 4.5 *Assume $\mu_1(dx) \in \mathcal{M}_{uni}(\mathbb{R}^d)$ satisfies (i) (or (ii)) of Lemma 4.2. Then for every $T > 0$ and $\lambda \in \mathbb{R}$ there exist finite constants $c > 0$ and $c_{\lambda,T} > 0$ such that*

(i) $|p_t(x,y) - p_{t'}(x,y)| \ \le\ c\int_t^{t'} \frac{1}{u} p_{2u}(x,y)du \quad \forall 0 < t \le t',\ x,y \in \mathbb{R}^d$

(ii) $\int_{\mathbb{R}^d} |p_t(x,y) - p_t(x',y)| e^{\lambda|y|}\mu_1(dy) \ \le\ c_{\lambda,T}\, t^{-(d/2+1/2-\alpha_1/2)}|x-x'|e^{\lambda|x|}e^{|\lambda||x-x'|}$

$\forall 0 < t \le T,\ x,x' \in \mathbb{R}^d$

(iii) $|p_t(x,y) - p_t(x',y)| \ \le\ c\, t^{-(d/2+1/2)}e^{-\frac{(|x'-y|-1)^2-1}{4t}}|x-x'| \quad \forall t > 0,\ |x-x'| \le 1,\ y \in \mathbb{R}^d.$

Proof Part (i) was proved in Lemme 2.1 of [Del96] and so we omit the proof. To prove part (ii) we note that for every $x, x', y \in \mathbb{R}^d$:

$$e^{-\frac{|x-y|^2}{2t}} = e^{-\frac{|\theta(x-x')+x'-y|^2}{2t}}\Big|_{\theta=1}, \qquad e^{-\frac{|x'-y|^2}{2t}} = e^{-\frac{|\theta(x-x')+x'-y|^2}{2t}}\Big|_{\theta=0} \qquad \text{and}$$

$$\begin{aligned}
\frac{d}{d\theta}e^{-\frac{|\theta(x-x')+x'-y|^2}{2t}} &= -\Big(\frac{d}{d\theta}\frac{|\theta(x-x')+x'-y|^2}{t}\Big)\, e^{-\frac{|\theta(x-x')+x'-y|^2}{2t}} \\
&= -\frac{\langle \theta(x-x')+x'-y, x-x'\rangle}{t}\, e^{-\frac{|\theta(x-x')+x'-y|^2}{2t}} \\
&\le -\frac{|\theta(x-x')+x'-y|\ |x-x'|}{t}\, e^{-\frac{|\theta(x-x')+x'-y|^2}{2t}}.
\end{aligned}$$

Using this together with the elementary inequality $he^{-h^2} \le c\, e^{-h^2/2}$ $(\forall h \ge 0)$ we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d} |p_t(x,y) - p_t(x',y)|e^{\lambda|y|}\mu_1(dy) \ = \ \int_{\mathbb{R}} \frac{1}{(2\pi t)^{d/2}}\Big|e^{-\frac{|x-y|^2}{2t}} - e^{-\frac{|x'-y|^2}{2t}}\Big|\, e^{\lambda|y|}\ \mu_1(dy) \\
&\le\ \int_{\mathbb{R}} \frac{1}{(2\pi t)^{d/2}} \int_0^1 \frac{|\theta(x-x')+x'-y|\ |x-x'|}{t}\, e^{-\frac{|\theta(x-x')+x'-y|^2}{2t}} d\theta\, e^{\lambda|y|}\ \mu_1(dy) \\
&\le\ c\,\frac{1}{t^{d/2+1/2}}\,|x-x'| \int_0^1 \int_{\mathbb{R}^d} e^{-\frac{|\theta(x-x')+x'-y|^2}{4t}} e^{\lambda|y|}\mu_1(dy)d\theta.
\end{aligned}$$

By means of Lemma 4.2$(i) \Rightarrow (iv)$ and Remark 4.3 the r.h.s. can easily be estimated by $c_T\, t^{-(d/2+1/2-\alpha/2)}\,|x-x'|e^{\lambda|\theta x+(1-\theta)x'|} \le c_T\, t^{-(d/2+1/2-\alpha/2)}\,|x-x'|e^{\lambda|x|}e^{|\lambda||x-x'|}$. This proves part (ii). To verify part (iii) we proceed similarly:

$$\begin{aligned}
|p_t(x,y) - p_t(x',y)| &\le\ c\,\frac{1}{t^{d/2+1/2}}\,|x-x'| \int_0^1 e^{-\frac{|\theta(x-x')+x'-y|^2}{4t}} d\theta \\
&=\ c\, t^{-(d/2+1/2)}\,|x-x'| \int_0^1 e^{-\frac{\theta|x-x'|^2 - 2\theta\langle x-x', y-x\rangle + |y-x'|^2}{4t}} d\theta \\
&\le\ c\, t^{-(d/2+1/2)}\,|x-x'| \int_0^1 e^{-\frac{|y-x'|^2 - 2|y-x'|}{4t}} d\theta \\
&=\ c\, t^{-(d/2+1/2)}\,|x-x'|\, e^{-\frac{(|y-x'|-1)^2-1}{4t}}
\end{aligned}$$

where we took the assumption $|x-x'| \le 1$ into account. □

For convenience set $p_t(.,.) :\equiv 0$ for all $t < 0$. In the next lemma we restrict to $d = 1$. Recall the definition of condition (A) from Definition 2.21.

Lemma 4.6 *Let $\mu(dtdx) \in \mathcal{M}([0,\infty)\times\mathbb{R})$ satisfy condition* (A) *with $\alpha_1, \alpha_2 \in [0,1]$ and set $\alpha := \alpha_1/2 + \alpha_2 - 1$. Then, for every $T > 0$ and $\lambda \geq 0$ there exists a finite constant $c_{\lambda,T} > 0$ such that for all $0 \leq t \leq t' \leq T$ and $x, x' \in \mathbb{R}$:*

$$\int_0^{t'}\int_{\mathbb{R}} \Big(p_{t-r}(x,y) - p_{t'-r}(x',y)\Big)^2 e^{\lambda|y|}\mu(drdy) \leq c_{\lambda,T}\Big(|t-t'|^{\alpha} + |x-x'|^{2\alpha}\Big)e^{\lambda|x|}e^{\lambda|x-x'|}.$$

Proof First of all recall Remark 4.3. For simplicity we only prove the case $\lambda = 0$. For the case $\lambda > 0$ we just had to use $(i) \Rightarrow (iv), (v)$ instead of $(i) \Rightarrow (ii)$ of Lemma 4.2. Clearly,

$$\begin{aligned}
&\int_0^{t'}\int_{\mathbb{R}} \Big(p_{t-r}(x,y) - p_{t'-r}(x',y)\Big)^2 \mu(drdy) \\
&\leq\ 2\Big\{ \int_0^{t'}\int_{\mathbb{R}} \Big(p_{t'-r}(x',y) - p_{t'-r}(x,y)\Big)^2 \mu(drdy) \\
&\quad + \int_0^{t}\int_{\mathbb{R}} \Big(p_{t'-r}(x,y) - p_{t-r}(x,y)\Big)^2 \mu(drdy) + \int_t^{t'}\int_{\mathbb{R}} p^2_{t'-r}(x,y)\mu(drdy)\Big\} \\
&=:\ 2\,\{\, I_1 + I_2 + I_3 \,\}.
\end{aligned}$$

In the remainder of the proof we are going to establish proper bounds for I_1, I_2 and I_3. Since $\mu(dtdx) = \mu_1(t,dx)\mu_2(dt)$ satisfies condition (A), we get by Lemma 4.2$(i) \Rightarrow (ii)$ and Lemma 4.4(i):

$$\begin{aligned}
I_3 &= \int_t^{t'} \frac{1}{2\pi(t'-r)}\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{t'-r}}\mu_1(r,dy)\mu_2(dr) \\
&\leq \frac{1}{2\pi}\int_t^{t'}\frac{1}{t'-r}\tilde{c}_T\,(t'-r)^{\alpha_1/2}\mu_2(dr)\ \leq\ \tilde{c}_T|t-t'|^{\alpha_1/2+\alpha_2-1}\ =\ \tilde{c}_T|t-t'|^{\alpha}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
I_2 \ =\ &\int_0^{t-|t-t'|}\int_{\mathbb{R}} \Big(p_{t'-r}(x,y) - p_{t-r}(x,y)\Big)^2 \mu_1(r,dy)\mu_2(dr) \\
&+ \int_{t-|t-t'|}^{t}\int_{\mathbb{R}} \Big(p_{t'-r}(x,y) - p_{t-r}(x,y)\Big)^2 \mu_1(r,dy)\mu_2(dr).
\end{aligned} \tag{4.1}$$

The second summand on the r.h.s. of (4.1) is bounded by

$$\begin{aligned}
\frac{1}{2\pi}\int_{t-|t-t'|}^{t}\Big(&\frac{1}{t-r}\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{t-r}}\mu_1(r,dy) \\
&+\frac{1}{t'-r}\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{t'-r}}\mu_1(r,dy) + \frac{1}{t-r}\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{2(t-r)}}\mu_2(r,dy)\Big)\mu_2(dr)
\end{aligned}$$

which can easily be estimated by $\tilde{c}_T|t-t'|^{\alpha_1/2+\alpha_2-1} = \tilde{c}_T|t-t'|^{\alpha}$ (proceed as for the estimate for I_3). Using Lemma 4.5(i), condition (A), Lemma 4.2$(i) \Rightarrow (ii)$ and Lemma

4.4 (ii), we obtain the following bound for the first summand on the r.h.s. of (4.1)

$$
\begin{aligned}
&\int_0^{t-|t-t'|}\int_{\mathbb{R}} c\Big(\int_{t-r}^{t'-r}\frac{1}{u}p_{2u}(x,y)du\Big)^2\mu_1(r,dy)\mu_2(dr)\\
\le\ & c^2\frac{1}{2\pi 2}\int_0^{t-|t-t'|}\Big(\int_{t-r}^{t'-r}\frac{1}{u^{3/2}}du\Big)^2\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{2(t'-r)}}\mu_1(r,dy)\mu_2(dr)\\
\le\ & c^2\frac{1}{2\pi 2}\int_0^{t-|t-t'|}\Big(\frac{1}{\sqrt{t-r}}-\frac{1}{\sqrt{t'-r}}\Big)^2\bar{c}_T(2(t'-r))^{\alpha_1/2}\mu_2(dr)\\
\le\ & \tilde{c}_T\int_0^{t-|t-t'|}\Big(\frac{\sqrt{|t-t'|}}{\sqrt{t-r}\sqrt{t'-r}}\Big)^2(t'-r)^{\alpha_1/2}\mu_2(dr)\\
\le\ & \tilde{c}_T|t-t'|\int_0^{t-|t-t'|}\frac{1}{t-r}\frac{1}{(t'-r)^{1-\alpha/2}}\mu_2(dr)\\
\le\ & \tilde{c}_T|t-t'|\int_0^{t-|t-t'|}\frac{1}{(t-r)^{2-\alpha_1/2}}\mu_2(dr)\\
\le\ & \tilde{c}'_T|t-t'|^{\alpha_1/2+\alpha_2-1} \ = \ \tilde{c}'_T|t-t'|^{\alpha}.
\end{aligned}
$$

We therefore have $I_2\le \tilde{c}''_T|t-t'|^{\alpha}$. If $|x-x'|^2\ge t'$, then I_1 can easily be bounded by $\tilde{c}''_T\, t'^{\alpha}\le \tilde{c}''_T\,|x-x'|^{2\alpha}$. If $|x-x'|^2<t'$, then we have

$$
\begin{aligned}
I_1\ \le\ & \int_{t'-|x-x'|^2}^{t'}\int_{\mathbb{R}} 2\Big(p^2_{t'-r}(x',y)+p^2_{t'-r}(x,y)\Big)\mu_1(r,dy)\mu_2(dr)\\
&+\int_0^{t'-|x-x'|^2}\int_{\mathbb{R}}\Big(p_{t'-r}(x',y)-p_{t'-r}(x,y)\Big)\Big)^2\mu_1(r,dy)\mu_2(dr).
\end{aligned}
\tag{4.2}
$$

Using condition (A), Lemma 4.2(i) $\Rightarrow$ (ii) and Lemma 4.4(i) as before, the first summand on the r.h.s. of (4.2) can be estimated by

$$
\begin{aligned}
&2\frac{1}{2\pi}\int_{t'-|x-x'|^2}^{t'}\frac{1}{t'-r}\int_{\mathbb{R}}\Big(e^{-\frac{(x'-y)^2}{t'-r}}+e^{-\frac{(x-y)^2}{t'-r}}\Big)\mu_1(r,dy)\mu_2(dr)\\
\le\ & \frac{1}{\pi}\int_{t'-|x-x'|^2}^{t'}\frac{1}{t'-r}2\bar{c}_T(t'-r)^{\alpha_1/2}\mu_2(dr)\ \le\ \tilde{c}_T|x-x'|^{2(\alpha_1/2+\alpha_2-1)}\ =\ \tilde{c}_T|x-x'|^{2\alpha}.
\end{aligned}
$$

The second summand on the r.h.s. of (4.2) is dominated by

$$
\begin{aligned}
&\int_0^{t'-|x-x'|^2}\frac{1}{\sqrt{2\pi(t'-r)}}\int_{\mathbb{R}}|p_{t'-r}(x',y)-p_{t'-r}(x,y)|\mu_1(r,dy)\mu_2(dr)\\
\le\ & \frac{1}{\sqrt{2\pi}}\int_0^{t'-|x-x'|^2}\frac{1}{(t'-r)^{1/2}}c'_T\frac{1}{(t'-r)^{1-\alpha_1/2}}|x-x'|\mu_2(dr)\\
\le\ & \frac{1}{\sqrt{2\pi}}c'_T|x-x'|\int_0^{t'-|x-x'|^2}\frac{1}{(t'-r)^{3/2-\alpha_1/2}}\mu_2(dr)\\
\le\ & \tilde{c}'_T|x-x'|^{1-2(3/2-\alpha_1/2-\alpha_2)}\ =\ \tilde{c}'_T|x-x'|^{2\alpha}
\end{aligned}
$$

where we applied (ii) of Lemma 4.5 and (ii) of Lemma 4.4. Hence, $I_1 \le \tilde{c}''_T |x-x'|^{2\alpha}$ and altogether, $2\{I_1+I_2+I_3\} \le c_T(|t-t'|^\alpha + |x-x'|^{2\alpha})$. $\square$

In the following lemma we turn back to arbitrary $d \ge 1$. Recall our convention $p_t(.,.) \equiv 0$ for $t<0$ and the definition of condition (B) from Definition 2.22.

Lemma 4.7 *Let $\mu(dtdx) \in \mathcal{M}([0,\infty)\times\mathbb{R}^d)$ satisfy condition* (B) *with $\beta_1 \in [0,d]$, $\beta_2 \in [0,1]$ and set $\beta := \beta_1/2+\beta_2-d/2$. Then, for every $T>0$ and $\lambda \ge 0$ there exists a finite constant $c_{\lambda,T}>0$ such that for all $0 \le t \le t' \le T$ and $x,x' \in \mathbb{R}^d$:*

$$\int_0^{t'}\int_{\mathbb{R}^d} \left|p_{t-r}(x,y)-p_{t'-r}(x',y)\right| e^{\lambda|y|}\mu(drdy) \;\le\; c_T\left(|t-t'|^\beta + |x-x'|^{2\beta}\right)e^{\lambda|x|}e^{\lambda|x-x'|}.$$

Proof As in the proof of Lemma 4.6 we restrict our attention to the case $\lambda=0$. The case $\lambda>0$ can be proved completely analogously. Now, recall Remark 4.3 and note

$$\begin{aligned}
&\int_0^{t'}\int_{\mathbb{R}^d} \left|p_{t-r}(x,y)-p_{t'-r}(x',y)\right|\mu(drdy)\\
&\le \int_0^{t'}\int_{\mathbb{R}^d} \left|p_{t'-r}(x',y)-p_{t'-r}(x,y)\right|\mu(drdy)\\
&\quad + \int_0^{t}\int_{\mathbb{R}^d} |p_{t-r}(x,y)-p_{t'-r}(x,y)|\,\mu(drdy) + \int_t^{t'}\int_{\mathbb{R}^d} p_{t'-r}(x,y)\mu(drdy)\\
&=: I_1+I_2+I_3.
\end{aligned}$$

As in the proof of Lemma 4.6 we shall establish proper bounds for I_1, I_2 and I_3. Since $\mu(dtdx)=\mu_1(t,dx)\mu_2(dt)$ satisfies condition (B), we get by Lemma 4.2$(i) \Rightarrow (ii)$ and Lemma 4.4(i):

$$\begin{aligned}
I_3 &= \int_t^{t'} \frac{1}{(2\pi(t'-r))^{d/2}}\int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{2(t'-r+)}}\mu_1(r,dy)\mu_2(dr)\\
&\le \frac{1}{(2\pi)^{d/2}}\int_t^{t'}\frac{1}{(t'-r)^{d/2}}c_T(2(t'-r'))^{\beta_1/2}\mu_2(dr)\\
&\le \tilde{c}_T\int_t^{t'}\frac{1}{(t'-r)^{d/2-\beta_1/2}}\mu_2(dr) \;\le\; c_T|t-t'|^{\beta_1/2+\beta_2-d/2} \;=\; c_T|t-t'|^\beta.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
I_2 = &\int_0^{t-|t-t'|}\int_{\mathbb{R}^d} |p_{t'-r'}(x,y)-p_{t-r'}(x,y)|\,\mu_1(r,dy)\mu_2(dr) \\
&+\int_{t-|t-t'|}^{t}\int_{\mathbb{R}^d} |p_{t'-r}(x,y)-p_{t-r}(x,y)|\,\mu_1(r,dy)\mu_2(dr).
\end{aligned} \tag{4.3}$$

The second summand on the r.h.s. of (4.3) is bounded by

$$\begin{aligned}
\frac{1}{(2\pi)^{d/2}}\int_{t-|t-t'|}^{t}\Big(&\frac{1}{(t'-r)^{d/2}}\int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{2(t'-r)}}\mu_1(r,dy)\\
&+\frac{1}{(t-r)^{d/2}}\int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{2(t-r)}}\mu_1(r,dy)\Big)\mu_2(dr)
\end{aligned}$$

which can easily be estimated by $\bar{c}_T|t-t'|^{\beta_1/2+\beta_2-d/2} = \tilde{c}_T|t-t'|^{\beta}$ (proceed as for the estimate for I_3). Using Lemma 4.5(i), condition (B), Lemma 4.2(i) $\Rightarrow$ (ii) and Lemma 4.4(ii), we obtain the following bound for the first summand on the r.h.s. of (4.3)

$$
\begin{aligned}
&\int_0^{t-|t-t'|}\int_{\mathbb{R}^d} c\int_{t-r}^{t'-r}\frac{1}{u}p_{2u}(x,y)du\mu_1(r,dy)\mu_2(dr)\\
&\quad= c\frac{1}{(2\pi)^{d/2}}\int_0^{t-|t-t'|}\int_{t-r}^{t'-r}\frac{1}{u^{1+d/2}}\int_{\mathbb{R}^d}e^{-\frac{|x-y|^2}{2u}}\mu_1(r,dy)du\mu_2(dr)\\
&\quad= c\frac{1}{(2\pi)^{d/2}}\int_0^{t-|t-t'|}\int_{t-r}^{t'-r}\frac{1}{u^{1+d/2}}c_T u^{\beta_1}du\mu_2(dr)\\
&\quad\le \frac{c}{(2\pi)^{d/2}}\int_0^{t-|t-t'|}\Big(\frac{1}{(t-r)^{d/2-\beta_1}}-\frac{1}{(t'-r)^{d/2-\beta_1}}\Big)\mu_2(dr)\\
&\quad\le \tilde{c}_T\int_0^{t-|t-t'|}\frac{|t-t'|^{d/2-\beta_1/2}}{(t-r)^{d/2-\beta_1/2}(t'-r)^{d/2-\beta_1/2}}\mu_2(dr)\\
&\quad\le \tilde{c}_T|t-t'|^{d/2-\beta_1/2}\int_0^{t-|t-t'|}\frac{1}{(t-r)^{d-\beta_1}}\mu_2(dr)\\
&\quad\le \tilde{c}_T|t-t'|^{d/2-\beta_1/2}\int_0^{t-|t-t'|}\frac{1}{(t-r)^{d-\beta_1}}\mu_2(dr)\\
&\quad\le \tilde{c}_T|t-t'|^{d/2-\beta_1/2}\frac{1}{|t-t'|^{d-\beta_1-\beta_2}}\\
&\quad= \bar{c}_T|t-t'|^{\beta_1/2+\beta_2-d/2} = \bar{c}_T|t-t'|^{\beta}.
\end{aligned}
$$

We therefore have $I_2 \le \bar{c}_T|t-t'|^{\beta}$. If $|x-x'|^2 \ge t'$, then I_1 can easily be bounded by $\bar{c}_T\, t'^{\beta} \le \bar{c}_T\, |x-x'|^{2\beta}$ (proceed as for the estimate for I_3). If $|x-x'|^2 < t'$, then we have

$$
\begin{aligned}
I_1 \le{}& \int_{t'-|x-x'|^2}^{t'}\int_{\mathbb{R}^d}\Big(p_{t'-r}(x',y)+p_{t'-r}(x,y)\Big)\mu_1(r,dy)\mu_2(dr)\\
&+\int_0^{t'-|x-x'|^2}\int_{\mathbb{R}^d}\big|p_{t'-r}(x',y)-p_{t'-r}(x,y)\big|\,\mu_1(r,dy)\mu_2(dr).
\end{aligned}
\tag{4.4}
$$

Using condition (B), Lemma 4.2(i) $\Rightarrow$ (ii) and Lemma 4.4(i) as before, the first summand on the r.h.s. of (4.4) can be estimated by

$$
\begin{aligned}
&\int_{t'-|x-x'|^2}^{t'}\frac{1}{(2\pi(t'-r))^{d/2}}\int_{\mathbb{R}^d}\Big(e^{-\frac{(x'-y)^2}{2(t'-r)}}+e^{-\frac{(x-y)^2}{2(t'-r)}}\Big)\mu_1(r,dy)\mu_2(dr)\\
&\qquad\le \frac{1}{(2\pi)^{d/2}}\int_{t'-|x-x'|^2}^{t'}\frac{1}{(t'-r)^{d/2}}2c_T(2(t'-r))^{\beta_1/2}\mu_2(dr)\\
&\qquad\le \tilde{c}_T\int_{t'-|x-x'|^2}^{t'}\frac{1}{(t'-r)^{d/2-\beta_1/2}}\mu_2(dr)\\
&\qquad\le \bar{c}_T|x-x'|^{2(\beta_1/2+\beta_2-d/2)} = \bar{c}_T|x-x'|^{2\beta}.
\end{aligned}
$$

The second summand on the r.h.s. of (4.4) is dominated by

$$\begin{aligned}
\int_0^{t'-|x-x'|^2} & \int_{\mathbb{R}^d} \left|p_{t'-r}(x',y) - p_{t'-r}(x,y)\right| \mu_1(r,dy)\mu_2(dr) \\
&\le \int_0^{t'-|x-x'|^2} \tilde{c}_T \frac{1}{(t'-r)^{1/2+d/2-\beta_1/2}} |x-x'| \mu_2(dr) \\
&\le \tilde{c}_T |x-x'| \int_0^{t'-|x-x'|^2} \frac{1}{(t'-r)^{1/2+d/2-\beta_1/2}} \mu_2(dr) \\
&\le \bar{c}_T |x-x'|^{1+2(-1/2-d/2+\beta_1/2+\beta_2)} \ = \ \bar{c}_T |x-x'|^{2\beta}
\end{aligned}$$

where we applied (ii) of Lemma 4.5 and (ii) of Lemma 4.4. Hence, $I_1 \le \bar{c}_T |x-x'|^{2\beta}$ and altogether, $I_1 + I_2 + I_3 \le c_T(|t-t'|^\beta + |x-x'|^{2\beta})$. □

Lemma 4.8 *Let $\mu(dtdx) \in \mathcal{M}([0,\infty) \times \mathbb{R}^d)$ satisfy condition* (B) *with $\beta_1 \in [0,d]$, $\beta_2 \in [0,1]$ and set $\beta := \beta_1/2 + \beta_2 - d/2$. Also, fix $\theta \in (0,\beta]$. Then, for every $T > 0$ there exists a finite constant $c_{\theta,T} > 0$ such that for all $0 \le s \le t \le T$, $x \in \mathbb{R}^d$ and $\epsilon, \epsilon' \in [0,1]$:*

$$\int_s^t \int_{\mathbb{R}^d} |p_{r+\epsilon}(x,y) - p_{r+\epsilon'}(x,y)| \, \mu(drdy) \ \le \ c_{\theta,T} \, (t-s)^{\beta-\theta} |\epsilon - \epsilon'|^\theta. \tag{4.5}$$

Proof Assume w.l.o.g. $\epsilon \le \epsilon'$. By means of Lemma 4.5(i), Lemma 4.2$(i) \Rightarrow (ii)$ and Lemma 4.4(i) we obtain for $\theta \in (0,\beta)$

$$\begin{aligned}
\int_s^t & \int_{\mathbb{R}^d} |p_{r+\epsilon}(x,y) - p_{r+\epsilon'}(x,y)| \, \mu(drdy) \\
&= \int_s^t \int_{\mathbb{R}^d} \int_{r+\epsilon}^{r+\epsilon'} \frac{1}{u} p_{2u}(x,y) du \, \mu(drdy) \\
&= \int_s^t \int_{r+\epsilon}^{r+\epsilon'} \frac{1}{u} \frac{1}{(4\pi u)^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4u}} \mu_1(r,dy) du \, \mu_2(dr) \\
&\le \int_s^t \int_{r+\epsilon}^{r+\epsilon'} \frac{1}{u} \frac{1}{(4\pi u)^{d/2}} c_T (4u)^{\beta_1/2} du \, \mu_2(dr) \\
&\le c'_T \int_s^t \frac{1}{r^{d/2-\beta_1/2+\theta}} \int_{r+\epsilon}^{r+\epsilon'} \frac{1}{u^{1-\theta}} du \, \mu_2(dr) \\
&\le c_{\theta,T} \, (t-s)^{\beta-\theta} |\epsilon - \epsilon'|^\theta.
\end{aligned}$$

If $\theta = \beta$, then proceed as in the proof of Lemma 4.7. □

Set $P_t f(x) := \int_{\mathbb{R}^d} p_t(x,y) f(y) dy$ for all $t > 0$, $x \in \mathbb{R}^d$ and $f \in C_{tem}(\mathbb{R}^d)$, where p denotes the heat kernel. The family $(P_t) \equiv (P_t)_{t \ge 0}$ provides a semigroup of bounded linear operators acting on functions. In particular, (P_t) on $C_0(\mathbb{R}^d)$ is a Feller semigroup whose generator is the d-dimensional Laplacian $\Delta = \sum_{i=1}^d \partial^2/\partial x_i^2$. Since (P_t) corresponds to the heat kernel, it is called *heat semigroup*.

Lemma 4.9 *If $\eta \in C_{tem}(\mathbb{R}^d)$ (resp. $\eta \in C_{rap}(\mathbb{R}^d)$), then $(P_t\eta(.) : t \geq 0)$ is $C_{tem}(\mathbb{R}^d)$-valued (resp. $C_{rap}(\mathbb{R}^d)$-valued) continuous. Moreover, we can find for every $t_0 > 0$ a finite constant $c_{t_0,\eta} > 0$ such that for all $\lambda > 0$ (resp. $\lambda < 0$), $t, t' \geq t_0$ and $x, x' \in \mathbb{R}^d$:*

$$\left|P_t\eta(x) - P_{t'}\eta(x')\right| \leq c_{t_0,\eta}\left(|t-t'|^{1/2} + |x-x'|\right) e^{\lambda|x|}e^{|\lambda||x-x'|}. \tag{4.6}$$

If $\eta \in C_b(\mathbb{R}^d)$ is Lipschitz continuous, then (4.6) holds for all $t, t' \geq 0$, $x, x' \in \mathbb{R}^d$, some finite constant $c_\eta > 0$ (instead of $c_{t_0,\eta}$) and $\lambda = 0$.

Proof First of all note that the following inequality holds for some constant $c > 0$:

$$\int_{\mathbb{R}^d} |x-y| e^{-\frac{|x-y|^2}{r}} dy \ \leq \ c\, r^{1/2+d/2} \qquad \forall r > 0,\ x \in \mathbb{R}^d. \tag{4.7}$$

We only show the case $\eta \in C_{tem}(\mathbb{R}^d)$; for $\eta \in C_{rap}(\mathbb{R}^d)$ one can proceed analogously. Let $\lambda > 0$. Then, for every $t, t' \geq 0$ and $x, x \in \mathbb{R}^d$,

$$\begin{aligned}
|P_t\eta(x) - P_{t'}\eta(x')| \ &\leq \ \int_{\mathbb{R}^d} |p_t(x,y) - p_t(x',y)|dy + \int_{\mathbb{R}^d} |p_t(x',y) - p_{t'}(x',y)|dy \\
&\leq \ |\eta|_{(-\lambda)} \int_{\mathbb{R}^d} |p_t(x,y) - p_t(x',y)|e^{\lambda|y|}dy + |\eta|_{(-\lambda)} \int_{\mathbb{R}^d} |p_t(x',y) - p_{t'}(x',y)|e^{\lambda|y|}dy.
\end{aligned}$$

Using Lemma 4.5(ii) we can estimate the first summand by $c_\eta t^{-1/2}|x-x'|e^{\lambda|x|}e^{\lambda|x-x'|}$. With help of Lemma 4.5$(i)$ and Lemma 4.2$(i) \Rightarrow (iv)$ the second summand can be estimated by $c_\eta(t \wedge t')^{-1/2}|t-t'|e^{\lambda|x'|}$. Hence, (4.6) holds. In particular, $(P_t\eta(.) : t \geq 0)$ is $C_{tem}(\mathbb{R}^d)$-valued continuous on $(0, \infty)$. Moreover, for every $t \geq 0$ and $x \in \mathbb{R}^d$,

$$\begin{aligned}
e^{-\lambda|x|}|P_t\eta(x) - \eta(x)| \ &= \ \Big|\int_{\mathbb{R}^d} p_t(x,y)(\eta(y) - \eta(x))e^{-\lambda|x|}dy\Big| \\
&\leq \ \Big|\int_{\mathbb{R}^d} p_t(x,y)(\eta(y)e^{-\lambda|y|} - \eta(x)e^{-\lambda|x|})dy\Big| + \Big|\int_{\mathbb{R}^d} p_t(x,y)\eta(y)(e^{-\lambda|x|} - e^{-\lambda|y|})dy\Big| \\
&\leq \ \left|P_t\left(\eta(.)e^{-\lambda|.|}\right)(x) - \eta(x)e^{-\lambda|x|}\right| + \int_{\mathbb{R}^d} p_t(x,y)\eta(y)|e^{-\lambda|x|} - e^{-\lambda|y|}|dy.
\end{aligned}$$

The first summand tends to 0 as $t \downarrow 0$ (uniformly in x) since $e^{-\lambda|.|}\eta(.) \in C_0(\mathbb{R}^d)$ and (P_t) is $\|.\|_\infty$-continuous on $C_0(\mathbb{R}^d)$. Using Hölder's inequality, (4.7) and techniques as in the proof of Lemma 4.2$(ii) \Rightarrow (iv)$, the second summand can be estimated by

$$\begin{aligned}
&\Big(\int_{\mathbb{R}^d} p_t(x,y)|e^{-\lambda|x|} - e^{-\lambda|y|}|dy\Big)^{1/2}\Big(\int_{\mathbb{R}^d} p_t(x,y)\eta^2(y)|e^{-\lambda|x|} - e^{-\lambda|y|}|dy\Big)^{1/2} \\
&\leq \ \Big(\int_{\mathbb{R}^d} p_t(x,y)|x-y|dy\Big)^{1/2}\Big(\int_{\mathbb{R}^d} p_t(x,y)\ |\eta|^2_{(-\lambda/2)}\ |e^{\lambda(|y|-|x|)} - 1|dy\Big)^{1/2} \\
&\leq \ \Big(c\,t^{1/2}\Big)^{1/2}\Big(c_\lambda|\eta|^2_{(-\lambda/2)}\Big)^{1/2},
\end{aligned}$$

and so it tends to 0 as $t \downarrow 0$ (uniformly in $x \in \mathbb{R}^d$), too. That is, $(P_t\eta(.) : t \geq 0)$ is also continuous at 0.

Now, let $\eta \in C_b(\mathbb{R}^d)$ be Lipschitz continuous and w.l.o.g. $t \leq t'$. Clearly,

$$|P_t\eta(x) - P_{t'}\eta(x')| \leq |P_t\eta(x) - P_{t'}\eta(x)| + |P_{t'}\eta(x) - P_{t'}\eta(x')|.$$

Since (P_t) is a $\|.\|_\infty$-contractive semigroup, we have $\|P_t\eta - P_{t'}\eta\|_\infty \leq \|\eta - P_{|t-t'|}\eta\|_\infty$. Further, using the Lipschitz continuity of η and (4.7), we get for all $0 \leq t \leq t'$ and $x \in \mathbb{R}^d$:

$$\begin{aligned}
&|\eta(x) - P_{|t-t'|}\eta(x)| \\
&\leq \int_{\mathbb{R}^d} |\eta(x) - \eta(y)| p_{|t-t'|}(x,y)dy \;\leq\; \frac{c}{|t-t'|^{d/2}} \int_{\mathbb{R}^d} L_\eta |x-y| e^{-\frac{|x-y|^2}{2|t-t'|}} dy \;\leq\; c_\eta |t-t'|^{1/2}.
\end{aligned}$$

Hence, $|P_t\eta(x) - P_{t'}\eta(x)| \leq \bar{c}|t-t'|^{1/2}$ for all $t, t' \geq 0$ and $x \in \mathbb{R}^d$. On the other hand, by means of a substitution $y = a + x' - x$ we get for all $t' \geq 0$ and $x, x' \in \mathbb{R}^d$:

$$\begin{aligned}
&|P_{t'}\eta(x) - P_{t'}\eta(x')| \\
&\leq \Big| \int_{\mathbb{R}^d} p_{t'}(x,y)\eta(y)dy - \int_{\mathbb{R}^d} p_{t'}(x', a + x' - x)\eta(a + x' - x)da \Big| \\
&\leq \int_{\mathbb{R}^d} p_{t'}(x,y)|\eta(y) - \eta(y + x' - x)|dy \\
&\leq \int_{\mathbb{R}^d} p_{t'}(x,y) L_\eta |y - (y + x' - x)|dy \;\leq\; L_\eta |x - x'|.
\end{aligned}$$

On the whole, $|P_t\eta(x) - P_{t'}\eta(x')| \leq c_\eta(|t-t'|^{1/2} + |x-x'|)$ for all $t, t' \geq 0$ and $x, x' \in \mathbb{R}^d$. □

4.3 Gronwall-type lemmas

We first recall the classical Gronwall lemma (cf., for instance, [KS91] p.288).

Lemma 4.10 [GRONWALL LEMMA] *Let $g : [0,\infty) \to \mathbb{R}$ be a continuous function and $T > 0$. Assume there exists a finite constant $c_T > 0$ and a Lebesgue integrable function $h : [0,T] \to \mathbb{R}$ such that*

$$g(t) \leq h(t) + c_T \int_0^t g(r)dr \qquad \forall t \leq T.$$

Then,

$$g(t) \leq h(t) + c_T \int_0^t h(r) e^{c_T(t-r)} dr \qquad \forall t \leq T.$$

In particular, if $h(t) \equiv c_0$ for some constant $c_0 \geq 0$, then

$$g(t) \leq c_0 e^{c_T t} \leq c_0 e^{c_T T} =: \tilde{c}_T\, c_0 \qquad \forall t \leq T.$$

Lemma 4.12 below partially improves the classical Gronwall lemma. On the one hand, we may consider a singular measure $\mu(dr)$ instead of dr, the integrand may have an additional singularity (whose maximal order is governed by the singularity of $\mu(dr)$) and g does not need to be continuous. On the other hand, the estimate's quality is reduced. For our purpose, however, the statement is completely sufficient. Modifications of the classical Gronwall lemma have been studied before. A standard generalization can be found, for instance, in [Wal96] (p.284). There the integrator is still the Lebesgue measure dr but the integrand looks different. For instance, it may have the additional singularity $(t-r)^{-\gamma}$ for any $\gamma \in (0,1)$.

Lemma 4.11 [Gronwall-type lemma] *Let $k \geq 1$ and $\mu_2^i(dt) \in \mathcal{M}([0,\infty))$ $(1 \leq i \leq k)$. Assume there exist $\alpha_2^i \in (0,1]$ $(1 \leq i \leq k)$ such that*

$$\forall T > 0\, \exists \hat{c}_T > 0 : \quad \sup_{t \leq T} \mu_2^i([0,\infty) \cap B[t,r]) \leq \hat{c}_T\, r^{\alpha_2^i} \quad \forall r \in (0,1], 1 \leq i \leq k.$$

Let $g_n : [0,\infty) \to [0,\infty)$ be measurable functions $(n \geq 1)$ and assume g_1 is bounded on compacts. Further, let $\gamma^i \in [0,\alpha_2^i)$ $(1 \leq i \leq k)$ and $c_0 \geq 0$ be constants. If for $T > 0$ there exists a constant $c_T > 0$ with

$$g_{n+1}(t) \leq c_T\Big(c_0 + \sum_{i=1}^k \sup_{s \leq t} \int_0^s \frac{1}{(s-r)^{\gamma^i}} g_n(r) \mu_2^i(dr)\Big) \quad \forall t \leq T, n \geq 1,$$

then there exist constants $q_T \in (0,1)$ and $\tilde{c}_T > 0$ (depending on T, c_T, $\hat{c}_T$, α_i, γ_i and $\|g_1\|_{\infty,T} := \sup_{t \leq T} g(t)$, and being independent of c_0) such that

$$\sup_{t \leq T} g_n(t) \;\leq\; \tilde{c}_T\;(\;c_0\;+\;q_T^n\;) \qquad \forall n \geq 1.$$

Proof First of all note that in Lemma 4.4(*iii*) the constant $c_T''' > 0$ is independent of $\delta > 0$. Set $\delta := \min\{\alpha_2^i - \gamma^i : 1 \leq i \leq k\} > 0$. By Lemma 4.4(*i*) and (*iii*) we can choose constants $c_T', c_T'' > 0$ such that for all $t \leq T$, $1 \leq i \leq k$, $j \geq 1$ and $\theta \in (0,1)$,

$$\sup_{s \leq t} \int_0^s \frac{1}{(s-r)^{\gamma^i}} \mu_2^i(dr) \;\leq\; c_T' t^{\alpha_2^i - \gamma^i} \;\leq\; c_T'' t^\delta, \tag{4.8}$$

$$\begin{aligned}\sup_{s \leq t} \int_0^s \frac{r^{j\delta}}{(s-r)^{\gamma^i}} \mu_2^i(dr) \;&\leq\; c_T' t^{j\delta + \alpha_2^i - \gamma^i} (\theta^{j\delta} + (1-\theta)^{\alpha_2^i - \gamma^i}) \\ &\leq\; c_T'' t^{(j+1)\delta} (\theta^{j\delta} + (1-\theta)^\delta).\end{aligned} \tag{4.9}$$

Set $\bar{c}_T := c_T'' \vee (c_T'' \|g_1\|_{\infty,T})$. By assumption and (4.8) we obtain for all $r_2 \in [0,T]$:

$$\begin{aligned} g_2(r_2) \;&\leq\; c_T\Big(c_0 + \sum_{i=1}^k \sup_{s_1 \leq r_2} \int_0^{s_1} \frac{1}{(s_1 - r_1)^{\gamma^i}} g_1(r_1) \mu_2^i(dr_1)\Big) \\ &\leq\; c_T\Big(c_0 + k\bar{c}_T r_2^\delta\Big) \;\leq\; c_0 c_T + c_T k \bar{c}_T r_2^\delta. \end{aligned}$$

Using this and again the assumption and (4.9), we obtain for all $r_3 \in [0,T]$:

$$\begin{aligned}
g_3(r_3) &\leq c_T\Big(c_0 + \sum_{i=1}^k \sup_{s_2 \leq r_3} \int_0^{s_2} \frac{1}{(s_2-r_2)^{\gamma^i}} g_2(r_2)\mu_2^i(dr_2)\Big) \\
&\leq c_T\Big(c_0 + \sum_{i=1}^k \sup_{s_2 \leq r_3} \int_0^{s_2} \frac{1}{(s_2-r_2)^{\gamma^i}} c_T(c_0 + k\bar{c}_T r_2^\delta)\mu_2^i(dr_2)\Big) \\
&\leq c_T\Big(c_0 + c_T c_0 \sum_{i=1}^k \sup_{s_2 \leq r_3} \int_0^{s_2} \frac{1}{(s_2-r_2)^{\gamma^i}} \mu_2^i(dr_2) \\
&\qquad + c_T k\bar{c}_T \sum_{i=1}^k \sup_{s_2 \leq r_3} \int_0^{s_2} \frac{r_2^\delta}{(s_2-r_2)^{\gamma^i}} \mu_2^i(dr_2)\Big) \\
&\leq c_T\Big(c_0 + c_T c_0 k\bar{c}_T r_3^\delta + c_T k^2 \bar{c}_T^2 r_3^{2\delta}(\theta^\delta + (1-\theta)^\delta)\Big) \\
&= c_0\Big[c_T + c_T^2 k\bar{c}_T r_3^\delta\Big] + c_T^2 k^2 \bar{c}_T^2 r_3^{2\delta}(\theta^\delta + (1-\theta)^\delta).
\end{aligned}$$

Using this and again the assumption and (4.9), we obtain for all $r_4 \in [0,T]$:

$$\begin{aligned}
g_4(r_4) &\leq c_T\Big(c_0 + \sum_{i=1}^k \sup_{s_3 \leq r_4} \int_0^{s_3} \frac{1}{(s_3-r_3)^{\gamma^i}} g_3(r_3)\mu_2^i(dr_3)\Big) \\
&\leq c_T\Big(c_0 + \sum_{i=1}^k \sup_{s_3 \leq r_4} \int_0^{s_3} \frac{1}{(s_3-r_3)^{\gamma^i}} \\
&\qquad \times c_T\Big(c_0 + c_T c_0 k\bar{c}_T r_3^\delta + c_T k^2 \bar{c}_T^2 r_3^{2\delta}(\theta^\delta + (1-\theta)^\delta)\Big)\mu_2^i(dr_3)\Big) \\
&\leq c_T\Big(c_0 + c_T c_0 \sum_{i=1}^k \sup_{s_3 \leq r_4} \int_0^{s_3} \frac{1}{(s_3-r_3)^{\gamma^i}} \mu_2^i(dr_3) \\
&\qquad + c_T^2 c_0 k\bar{c}_T \sum_{i=1}^k \sup_{s_3 \leq r_4} \int_0^{s_3} \frac{r_3^\delta}{(s_3-r_3)^{\gamma^i}} \mu_2^i(dr_3) \\
&\qquad + c_T^2 k^2 \bar{c}_T^2 (\theta^\delta + (1-\theta)^\delta) \sum_{i=1}^k \sup_{s_3 \leq r_4} \int_0^{s_3} \frac{r_3^{2\delta}}{(s_3-r_3)^{\gamma^i}} \mu_2^i(dr_3)\Big) \\
&\leq c_T\Big(c_0 + c_T c_0 k\bar{c}_T r_4^\delta + c_T^2 c_0 k^2 \bar{c}_T{}^2 r_4^{2\delta}(\theta^\delta + (1-\theta)^\delta) \\
&\qquad + c_T^2 k^3 \bar{c}_T^3 r_4^{3\delta}(\theta^{2\delta} + (1-\theta)^\delta)(\theta^\delta + (1-\theta)^\delta)\Big) \\
&= c_0\Big[c_T + c_T^2 k\bar{c}_T r_4^\delta + c_T^3 k^2 \bar{c}_T{}^2 r_4^{2\delta}(\theta^\delta + (1-\theta)^\delta)\Big] \\
&\qquad + c_T^3 k^3 \bar{c}_T^3 r_4^{3\delta}(\theta^{2\delta} + (1-\theta)^\delta)(\theta^\delta + (1-\theta)^\delta).
\end{aligned}$$

Going ahead recursively, we obtain for all $n \geq 1$ and $r_n \in [0,T]$:

$$g_n(r_n) \leq c_T\Big(c_0 + \sum_{i=1}^k \sup_{s_{n-1} \leq r_n} \int_0^{s_{n-1}} \frac{1}{(s_{n-1}-r_{n-1})^{\gamma^i}} g_{n-1}(r_{n-1})\mu_2^i(dr_{n-1})\Big) \leq$$

$$
\begin{aligned}
&c_0\Big[c_T + c_T^2 k\bar{c_T} r_n^\delta + c_T^3 k^2 \bar{c_T}^2 r_n^{2\delta}\Big(\theta^\delta + (1-\theta)^\delta\Big) + \\
&\quad c_T^4 k^3 \bar{c_T}^3 r_n^{3\delta}\Big(\theta^{2\delta} + (1-\theta)^\delta\Big)\Big(\theta^\delta + (1-\theta)^\delta\Big) + \cdots + \\
&\quad c_T^{n-1} k^{n-2} \bar{c_T}^{n-2} r_n^{(n-2)\delta}\Big(\theta^{(n-2)\delta} + (1-\theta)^\delta\Big)\Big(\theta^{(n-3)\delta} + (1-\theta)^\delta\Big)\cdots\Big(\theta^\delta + (1-\theta)^\delta\Big)\Big] \\
&+ c_T^{n-1} k^{n-1} \bar{c}_T^{n-1} r_n^{(n-1)\delta}\Big(\theta^{(n-1)\delta} + (1-\theta)^\delta\Big)\Big(\theta^{(n-2)\delta} + (1-\theta)^\delta\Big)\cdots\Big(\theta^\delta + (1-\theta)^\delta\Big).
\end{aligned}
$$

Setting $K_T := c_T k \bar{c}_T T^\delta$ yields for every $n \geq 1$ and $r_n \in [0,T]$:

$$
\begin{aligned}
g_n(r_n) \quad \leq \quad & c_0\Big[c_T + c_T K_T + c_T K_T^2\Big(\theta^\delta + (1-\theta)^\delta\Big) + \ldots + \\
& \qquad c_T K_T^{n-2}\Big(\theta^{(n-2)\delta} + (1-\theta)^\delta\Big)\Big(\theta^{(n-3)\delta} + (1-\theta)^\delta\Big)\cdots\Big(\theta^\delta + (1-\theta)^\delta\Big)\Big] \\
& + K_T^{n-1}\Big(\theta^{(n-1)\delta} + (1-\theta)^\delta\Big)\Big(\theta^{(n-2)\delta} + (1-\theta)^\delta\Big)\cdots\Big(\theta^\delta + (1-\theta)^\delta\Big).
\end{aligned}
\tag{4.10}
$$

Pick $\epsilon \in (0, K_T^{-1} \wedge 2)$, set $\theta = 1 - (\epsilon/2)^{1/\delta}$ and choose $j_\epsilon \geq 1$ in such a manner that $\theta^{j\delta} \leq \epsilon/2$ holds for all $j \geq j_\epsilon$. Thus $(\theta^{j\delta} + (1-\theta)^\delta) \leq \epsilon$ holds for all $j \geq j_\epsilon$. Set $M_\epsilon = \sup_{j=1,\ldots,j_\epsilon-1}(\theta^{j\delta} + (1-\theta)^\delta)(\theta^{(j-1)\delta} + (1-\theta)^\delta)\cdots(\theta^\delta + (1-\theta)^\delta)\epsilon^{-(j-1)}$ and define $q_{\epsilon,T} = \epsilon K_T$ $(\in (0,1))$. Then we obtain by (4.10):

$$
\begin{aligned}
g_n(r_n) \quad & \leq \quad c_T c_0\Big[1 + K_T + K_T^2 M_\epsilon \epsilon^2 + \ldots + K_T^{n-2} M_\epsilon \epsilon^{n-2}\Big] + K_T^{n-1} M_\epsilon \epsilon^{n-1} \\
& \leq \quad c_{\epsilon,T}\Big(c_0\Big[1 + K_T + K_T^2 \epsilon^2 + \ldots + K_T^{n-2}\epsilon^{n-2}\Big] + K_T^{n-1}\epsilon^{n-1}\Big) \\
& \leq \quad c_{\epsilon,T}\Big(c_0\Big[1 + K_T + q_{\epsilon,T}^2 + \ldots + q_{\epsilon,T}^{n-2}\Big] + q_{\epsilon,T}^{n-1}\Big) \ \leq\ \tilde{c}_{\epsilon,T}\Big(c_0 + q_{\epsilon,T}^n\Big)
\end{aligned}
$$

for all $r_n \in [0,T]$ and $n \geq 1$. □

As an immediate consequence of Lemma 4.11 we obtain

Lemma 4.12 [Gronwall-type lemma] *If $g_n = g$ ($\forall n \geq 1$) in the setting of Lemma 4.11, then*

$$
g(t) \leq c_T\Big(c_0 + \sum_{i=1}^k \sup_{s\leq t}\int_0^s \frac{1}{(s-r)^{\gamma^i}} g(r)\mu_2^i(dr)\Big) \qquad \forall t \leq T, n \geq 1
$$

implies $\sup_{t\leq T} g(t) \leq \tilde{c}_T c_0$.

The following two lemmas are generalizations of a result of Shiga ([Shi94], Lemma 6.4(ii)).

Lemma 4.13 *Let $\varrho(dtdx), \sigma(dtdx) \in \mathcal{M}([0,\infty)\times\mathbb{R})$ satisfy condition* (A)*, resp.* (B)*, and fix $\lambda > 0$. Assume $g, h : [0,\infty)\times\mathbb{R} \to [0,\infty)$ are measurable functions satisfying*

$$
\sup_{t\leq T}\sup_{x\in\mathbb{R}} e^{-\lambda|x|} g(t,x) < \infty \qquad \text{and} \qquad \sup_{t\leq T}\sup_{x\in\mathbb{R}} e^{+\lambda|x|} h(t,x) < \infty \tag{4.11}
$$

for some $T>0$. If for the same $T>0$ there exists a constant $c_T>0$ such that

$$\begin{aligned} g(t,x) \quad &\leq \quad c_T\Big(h(t,x)+\sup_{s\leq t}\int_0^s\int_{\mathbb{R}} p_{s-r}(x,y)g(r,y)\sigma(drdy) \\ &\qquad + \sup_{s\leq t}\int_0^s\int_{\mathbb{R}} p_{s-r}^2(x,y)g(r,y)\varrho(drdy)\Big) \end{aligned} \tag{4.12}$$

holds for all $t\leq T$ and $x\in\mathbb{R}$, then: $\quad \sup_{t\leq T}\sup_{x\in\mathbb{R}} e^{+\lambda|x|}g(t,x)<\infty$.

Proof As already mentioned, the constant $c_T'''>0$ in (iii) of Lemma 4.4 is independent of $\delta>0$. Set $\delta:=\min\{\frac{\alpha_1}{2}+\alpha_2-1,\frac{\beta_1}{2}+\beta_2-\frac{1}{2}\}>0$. By Lemma 4.2$(i)\Rightarrow(ii)$ and $(i)\Rightarrow(iv)$ and Lemma 4.4(i) and (iii) we can choose a constant $\bar{c}_T=\bar{c}_{\lambda,T}>0$ such that for all $t\leq T$, $j\geq 1$, $\theta\in(0,1)$ and $\Lambda\in\{\lambda;0\}$:

$$\begin{aligned} \sup_{s\leq t}\int_0^s\int_{\mathbb{R}} e^{\pm\Lambda|y|}p_{s-r}(x,y)\sigma(drdy) \quad &\leq \quad \bar{c}_T\, e^{\pm\Lambda|x|}\, t^{\delta}, \\ \sup_{s\leq t}\int_0^s\int_{\mathbb{R}} e^{\pm\Lambda|y|}p_{s-r}^2(x,y)\varrho(drdy) \quad &\leq \quad \bar{c}_T\, e^{\pm\Lambda|y|}\, t^{\delta}, \\ \sup_{s\leq t}\int_0^s\int_{\mathbb{R}} e^{\pm\Lambda|y|}p_{s-r}(x,y)r^{j\delta}\sigma(drdy) \quad &\leq \quad \bar{c}_T\, e^{\pm\Lambda|x|}\,(\theta^{j\delta}+(1-\theta)^{\theta})\, t^{(j+1)\delta}, \\ \sup_{s\leq t}\int_0^s\int_{\mathbb{R}} e^{\pm\Lambda|y|}p_{s-r}^2(x,y)r^{j\delta}\varrho(drdy) \quad &\leq \quad \bar{c}_T\, e^{\pm\Lambda|x|}\,(\theta^{j\delta}+(1-\theta)^{\theta})\, t^{(j+1)\delta}. \end{aligned} \tag{4.13}$$

By (4.11) there exists a constant $C_T>0$ such that $\sup_{t\leq T}g(t,x)\leq C_Te^{+\lambda|x|}$ for all $x\in\mathbb{R}$. Then, using (4.12) and (4.13), we obtain for $r_2\in[0,T]$ and $x\in\mathbb{R}$:

$$\begin{aligned} g(r_2,x) \quad &\leq \quad c_T\Big(h(r_2,x)+\sup_{s_1\leq r_2}\int_0^{s_1}\int_{\mathbb{R}} p_{s_1-r_1}(x,y)g(r_1,y)\sigma(dr_1dy) \\ &\qquad + \sup_{s_1\leq r_2}\int_0^{s_1}\int_{\mathbb{R}} p_{s_1-r_1}^2(x,y)g(r_1,y)\varrho(dr_1dy)\Big) \\ &\leq \quad c_T\Big(h(r_2,x)+\sup_{s_1\leq r_2}\int_0^{s_1}\int_{\mathbb{R}} p_{s_1-r_1}(x,y)C_Te^{\lambda|y|}\sigma(dr_1dy) \\ &\qquad + \sup_{s_1\leq r_2}\int_0^{s_1}\int_{\mathbb{R}} p_{s_1-r_1}^2(x,y)C_Te^{\lambda|y|}\varrho(dr_1dy)\Big) \\ &\leq \quad c_T\Big(h(r_2,x)+2C_T\bar{c}_Te^{\lambda|x|}r_2^{\delta}\Big). \end{aligned}$$

Using this, the definition $\bar{C}_T:=\bar{c}_T\vee(C_T\bar{c}_T)$ and again (4.12) and (4.13), we obtain for $r_3\in[0,T]$ and $x\in\mathbb{R}$:

$$\begin{aligned} g(r_3,x) \quad &\leq \quad c_T\Big(h(r_3,x)+\sup_{s_2\leq r_3}\int_0^{s_2}\int_{\mathbb{R}} p_{s_2-r_2}(x,y)g(r_2,y)\sigma(dr_2dy) \\ &\qquad + \sup_{s_2\leq r_3}\int_0^{s_2}\int_{\mathbb{R}} p_{s_2-r_2}^2(x,y)g(r_2,y)\varrho(dr_2dy)\Big) \end{aligned}$$

$$
\begin{aligned}
&\le c_T\Big(h(r_3,x)+\sup_{s_2\le r_3}\int_0^{s_2}\int_{\mathbb{R}} p_{s_2-r_2}(x,y)c_T\Big(h(r_2,y)+2\bar{C}_Te^{\lambda|y|}r_2^\delta\Big)\sigma(dr_2dy)\\
&\qquad+\sup_{s_2\le r_3}\int_0^{s_2}\int_{\mathbb{R}} p^2_{s_2-r_2}(x,y)c_T\Big(h(r_2,y)+2\bar{C}_Te^{\lambda|y|}r_2^\delta\Big)\varrho(dr_2dy)\Big)\\
&\le c_T\Big(h(r_3,x)+c_T\|he^{+\lambda|.|}\|_\infty\sup_{s_2\le r_3}\int_0^{s_2}\int_{\mathbb{R}} e^{-\lambda|y|}p_{s_2-r_2}(x,y)\sigma(dr_2dy)\\
&\qquad+2c_T\bar{C}_T\sup_{s_2\le r_3}\int_0^{s_2}\int_{\mathbb{R}} e^{\lambda|y|}p_{s_2-r_2}(x,y)r_2^\delta\sigma(dr_2dy)\\
&\qquad+c_T\|he^{+\lambda|.|}\|_\infty\sup_{s_2\le r_3}\int_0^{s_2}\int_{\mathbb{R}} e^{-\lambda|y|}p^2_{s_2-r_2}(x,y)\varrho(dr_2dy)\\
&\qquad+2c_T\bar{C}_T\sup_{s_2\le r_3}\int_0^{s_2}\int_{\mathbb{R}} e^{\lambda|y|}p^2_{s_2-r_2}(x,y)r_2^\delta\varrho(dr_2dy)\Big)\\
&\le c_T\Big(h(r_3,x)+c_T2\bar{C}_T\|he^{+\lambda|.|}\|_\infty e^{-\lambda|x|}r_3^\delta+c_T(2\bar{C}_T)^2(\theta^\delta+(1-\theta)^\theta)e^{\lambda|x|}r_3^{2\delta}\Big).
\end{aligned}
$$

Using this and again (4.12) and (4.13), we obtain for $r_4\in[0,T]$ and $x\in\mathbb{R}$:

$$
\begin{aligned}
&g(r_4,x)\\
&\le c_T\Big(h(r_4,x)+\sup_{s_3\le r_4}\int_0^{s_3}\int_{\mathbb{R}} p_{s_3-r_3}(x,y)g(r_3,y)\sigma(dr_3dy)\\
&\qquad+\sup_{s_3\le r_4}\int_0^{s_3}\int_{\mathbb{R}} p^2_{s_3-r_3}(x,y)g(r_3,y)\varrho(dr_3dy)\Big)\\
&\le c_T\Big(h(r_4,x)\\
&\qquad+\sup_{s_3\le r_4}\int_0^{s_3}\int_{\mathbb{R}} p_{s_3-r_3}(x,y)c_T\Big[h(r_3,x)+c_T2\bar{C}_T\|he^{+\lambda|.|}\|_\infty e^{-\lambda|y|}r_3^\delta\\
&\qquad\qquad+c_T(2\bar{C}_T)^2(\theta^\delta+(1-\theta)^\theta)e^{\lambda|y|}r_3^{2\delta}\Big]\sigma(dr_3dy)\\
&\qquad+\sup_{s_3\le r_4}\int_0^{s_3}\int_{\mathbb{R}} p^2_{s_3-r_3}(x,y)c_T\Big[h(r_3,x)+c_T2\bar{C}_T\|he^{+\lambda|.|}\|_\infty e^{-\lambda|y|}r_3^\delta\\
&\qquad\qquad+c_T(2\bar{C}_T)^2(\theta^\delta+(1-\theta)^\theta)e^{\lambda|y|}r_3^{2\delta}\Big]\varrho(dr_3dy)\Big)\\
&\le c_T\Big(h(r_4,x)+\sup_{s_3\le r_4}\int_0^{s_3}\int_{\mathbb{R}} p_{s_3-r_3}(x,y)g(r_3,y)\sigma(dr_3dy)\\
&\qquad+\sup_{s_3\le r_4}\int_0^{s_3}\int_{\mathbb{R}} p^2_{s_3-r_3}(x,y)g(r_3,y)\varrho(dr_3dy)\Big)\\
&\le c_T\Big(h(r_4,x)\\
&\qquad+c_T\|he^{+\lambda|.|}\|_\infty\sup_{s_3\le r_4}\int_0^{s_3}\int_{\mathbb{R}} e^{-\lambda|y|}p_{s_3-r_3}(x,y)\sigma(dr_3dy)\\
&\qquad+c_T2\bar{C}_T\|he^{+\lambda|.|}\|_\infty\sup_{s_3\le r_4}\int_0^{s_3}\int_{\mathbb{R}} e^{-\lambda|y|}p_{s_3-r_3}(x,y)r_3^\delta\sigma(dr_3dy)
\end{aligned}
$$

$$
\begin{aligned}
&\quad +c_T(2\bar C_T)^2(\theta^\delta+(1-\theta)^\theta)\sup_{s_3\le r_4}\int_0^{s_3}\int_{\mathbb{R}} e^{\lambda|y|}p_{s_3-r_3}(x,y)r_3^{2\delta}\sigma(dr_3dy)\\
&+c_T\|he^{+\lambda|.|}\|_\infty\sup_{s_3\le r_4}\int_0^{s_3}\int_{\mathbb{R}} e^{-\lambda|y|}p^2_{s_3-r_3}(x,y)\varrho(dr_3dy)\\
&\quad +c_T2\bar C_T\|he^{+\lambda|.|}\|_\infty\sup_{s_3\le r_4}\int_0^{s_3}\int_{\mathbb{R}} e^{-\lambda|y|}p^2_{s_3-r_3}(x,y)r_3^{\delta}\varrho(dr_3dy)\\
&\quad +c_T(2\bar C_T)^2(\theta^\delta+(1-\theta)^\theta)\sup_{s_3\le r_4}\int_0^{s_3}\int_{\mathbb{R}} e^{\lambda|y|}p^2_{s_3-r_3}(x,y)r_3^{2\delta}\varrho(dr_3dy)\\
\le\ & c_T\Big(h(r_4,x)\\
&+c_T2\bar C_T\|he^{+\lambda|.|}\|_\infty e^{-\lambda|x|}r_4^\delta\\
&\quad +c_T(2\bar C_T)^2\|he^{+\lambda|.|}\|_\infty(\theta^\delta+(1-\theta)^{2\theta})e^{-\lambda|x|}r_4^{2\delta}\\
&\quad +c_T(2\bar C_T)^3(\theta^{2\delta}+(1-\theta)^\theta)(\theta^\delta+(1-\theta)^{3\theta})e^{\lambda|x|}r_4^{3\delta}\Big).
\end{aligned}
$$

Going ahead recursively, we obtain for all $n\ge 1$ and $r_n\in[0,T]$:

$$
\begin{aligned}
&g(r_n,x)\\
&\le\ c_T\Big(h(r_n,x)+\sup_{s_{n-1}\le r_n}\int_0^{s_{n-1}}\int_{\mathbb{R}} p_{s_{n-1}-r_{n-1}}(x,y)g(r_{n-1},y)\sigma(dr_{n-1}dy)\\
&\qquad\qquad +\sup_{s_{n-1}\le r_n}\int_0^{s_{n-1}}\int_{\mathbb{R}} p^2_{s_{n-1}-r_{n-1}}(x,y)g(r_{n-1},y)\varrho(dr_{n-1}dy)\Big)\\
&\le\ c_T\Big[h(r_n,x)\\
&\quad +c_T2\bar C_T\|he^{+\lambda|.|}\|_\infty e^{-\lambda|x|}r_n^\delta\\
&\qquad +c_T(2\bar C_T)^2\|he^{+\lambda|.|}\|_\infty\Big(\theta^\delta+(1-\theta)^\theta\Big)e^{-\lambda|x|}r_n^{2\delta}\Big)\\
&\qquad +c_T^2(2\bar C_T)^3\|he^{+\lambda|.|}\|^2_\infty\Big(\theta^{2\delta}+(1-\theta)^\theta\Big)\Big(\theta^\delta+(1-\theta)^\theta\Big)e^{-\lambda|x|}r_n^{3\delta}\\
&\qquad \cdots\\
&\qquad +c_T^{n-2}(2\bar C_T)^{n-1}\|he^{+\lambda|.|}\|^{n-2}_\infty\\
&\qquad\quad \times\Big(\theta^{(n-2)\delta}+(1-\theta)^\theta\Big)\Big(\theta^{(n-3)\delta}+(1-\theta)^\theta\Big)\cdots\Big(\theta^\delta+(1-\theta)^\theta\Big)e^{-\lambda|x|}r_n^{(n-2)\delta}\\
&\qquad +c_T^{n-1}(2\bar C_T)^{n-1}\\
&\qquad\quad \times\Big(\theta^{(n-1)\delta}+(1-\theta)^\theta\Big)\Big(\theta^{(n-2)\delta}+(1-\theta)^\theta\Big)\cdots\Big(\theta^\delta+(1-\theta)^\theta\Big)e^{\lambda|x|}r_n^{(n-1)\delta}\Big].
\end{aligned}
$$

Arguing as at the end of the proof of Lemma 4.11, we deduce

$$
\begin{aligned}
g(t,x)\ &\le\ \tilde c'_T\Big(\ h(t,x)\ +\ e^{-\lambda|x|}\Big[1+q+q^2+q^3+\ldots+q^{n-1}\Big]+q^ne^{+\lambda|x|}\Big)\\
&\le\ \tilde c''_T\Big(\ h(t,x)\ +\ e^{-\lambda|x|}\ +\ q^ne^{+\lambda|x|}\Big)\qquad \forall t\le T, x\in\mathbb{R}, n\ge 1
\end{aligned}
$$

for some constants $q=q_T\in(0,1)$, $\tilde c'_T,\tilde c''_T>0$ (depending on c_T, $\bar C_T$, $\|he^{+\lambda|.|}\|_\infty$ and δ). Letting $n\to\infty$ yields

$$
g(t,x)\ \le\ \tilde c''_T\ \Big(h(t,x)\ +\ e^{-\lambda|x|}\Big)
$$

for all $t \leq T$ and $x \in \mathbb{R}$. Then the claim follows from (4.11). $\square$

Lemma 4.14 *Let* $d \geq 1$, $\sigma(dtdx) \in \mathcal{M}([0,\infty) \times \mathbb{R}^d)$ *satisfy condition* (B) *and* $\lambda > 0$. *Assume* $g, h : [0,\infty) \times \mathbb{R}^d \to [0,\infty)$ *are measurable functions satisfying*

$$\sup_{t \leq T} \sup_{x \in \mathbb{R}^d} e^{-\lambda|x|} g(t,x) < \infty \quad \text{and} \quad \sup_{t \leq T} \sup_{x \in \mathbb{R}^d} e^{+\lambda|x|} h(t,x) < \infty$$

for some $T > 0$. *If for the same* $T > 0$ *there exists a constant* $c_T > 0$ *such that*

$$g(t,x) \leq c_T \Big(h(t,x) + \sup_{s \leq t} \int_0^s \int_{\mathbb{R}^d} p_{s-r}(x,y) g(r,y) \sigma(drdy) \Big)$$

holds for all $t \leq T$ *and* $x \in \mathbb{R}^d$, *then:* $\sup_{t \leq T} \sup_{x \in \mathbb{R}^d} e^{+\lambda|x|} g(t,x) < \infty$.

Proof Mimic the proof of Lemma 4.13! $\square$

5 Notion of stochastic differential equations

The main intension of this chapter is to give a sensible definition for jointly continuous solutions of the stochastic heat equation (1.2). This will be done in Definition 5.20. We shall adopt Walsh's notion of stochastic partial differential equations (SPDEs) involving stochastic integrals against orthogonal martingale measures ([Wal86]). In Section 5.3 we thoroughly review the crucial parts of Walsh's integration theory and we add some results. Before, in Section 5.2, we recall basics of Itô's stochastic calculus for continuous semimartingales. We thereby make this thesis more self-contained; later on we will also deal with Itô-integrals. In the Introduction (Chapter 1) we already discussed the relation of white noises to some Gaussian set functions which we refer to as *white noise measures.* Section 5.1 will be devoted to that sort of "measures". White noise measures can be identified with the integrators of certain Itô- (resp. Walsh-)integrals that are used for the definition of solutions to stochastic ordinary (resp. partial) differential equations driven by white noises. This will also be explained in Section 5.2 (resp. 5.3).

5.1 White noise measures

We start with an analysis of white noise measures. In this context the word *measure* is rather vague since white noise measures are in general merely additive set functions rather than σ-additive signed measures. Let $m \geq 1$ and recall that $\mathcal{A}(\mathbb{R}^m)$ denotes the algebra of relatively compact set in $\mathbb{R}^m$.

Definition 5.1 [WHITE NOISE MEASURE] *Let $\varrho(dx) \in \mathcal{M}(\mathbb{R}^m)$. A $\mathbb{R}$-valued process $\bar{W}^\varrho = (\bar{W}^\varrho(A) : A \in \mathcal{A}(\mathbb{R}^m))$ on some $[\Omega, \mathcal{F}, \mathbb{P}]$ is called white noise measure based on $\varrho(dx)$ if:*

(i) $\bar{W}^\varrho(A) \sim N(0, \varrho(A))$ for all $A \in \mathcal{A}(\mathbb{R}^m)$,

(ii) $\bar{W}^\varrho(A)$ and $\bar{W}^\varrho(A')$ are independent for any disjoint $A, A' \in \mathcal{A}(\mathbb{R}^m)$,

(iii) $\bar{W}^\varrho(A \cup A') = \bar{W}^\varrho(A) + \bar{W}^\varrho(A')$ $\mathbb{P}$-almost surely, for any disjoint $A, A' \in \mathcal{A}(\mathbb{R}^m)$.

The measure $\varrho(dx)$ is called *intensity measure* of $\bar{W}^\varrho$. Let us show that the white noise measure exists. Set $\Gamma(A, A') := \varrho(A \cap A')$ for every $A, A' \in \mathcal{A}(\mathbb{R}^m)$. Then Γ is obviously symmetric, and positive definite since

$$\sum_{i,j=1}^{k} \lambda_i \lambda_j \Gamma(A_i, A_j) = \sum_{i,j=1}^{k} \lambda_i \lambda_j \int_{\mathbb{R}^m} \mathbf{1}_{A_i}(x) \mathbf{1}_{A_j}(x) \varrho(dx) = \int_{\mathbb{R}^m} \Big(\sum_{i=1}^{k} \lambda_i \mathbf{1}_{A_i}(x) \Big)^2 \varrho(dx) \geq 0$$

holds for all $A_1, \ldots, A_k \in \mathcal{A}(\mathbb{R}^m)$ and $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$. Hence, by Proposition 3.18 there exists a centered Gaussian process $\bar{W}^\varrho = [(\bar{W}^\varrho(A) : A \in \mathcal{A}(\mathbb{R}^m)), \Omega, \mathcal{F}, \mathbb{P}]$ with covariation function Γ. This process is a white noise measure based on $\varrho(dx)$ since it satisfies $(i)-(iii)$ of Definition 5.1. Condition (i) is obvious, (ii) follows from the fact that two uncorrelated Gaussian random variables are independent, and (iii) is a consequence of

$$\begin{aligned} & \mathbb{E}[\{\bar{W}^\varrho(A) + \bar{W}^\varrho(A') - \bar{W}^\varrho(A \cup A')\}^2] \\ = \; & \mathbb{E}[\bar{W}^\varrho(A)^2 + \bar{W}^\varrho(A')^2 + \bar{W}^\varrho(A \cup A')^2 + 2\bar{W}^\varrho(A)\bar{W}^\varrho(A') \end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad -2\bar{W}^\varrho(A)\bar{W}^\varrho(A\cup A') - 2\bar{W}^\varrho(A')\bar{W}^\varrho(A\cup A')] \\
&= \varrho(A) + \varrho(A') + \varrho(A\cup A') + 0 - 2\varrho(A\cap(A\cup A')) - 2\varrho(A'\cap(A\cup A')) \;=\; 0
\end{aligned}
$$

(for all disjoint $A, A' \in \mathcal{A}(\mathbb{R}^m)$). Conversely, let $\bar{W}^\varrho$ be a white noise measure. Then the sum $\sum_{i=1}^k \lambda_i \bar{W}^\varrho(A_i)$ is normally distributed[17] for every $k \geq 1$, $A_1, \ldots A_k \in \mathcal{A}(\mathbb{R}^m)$ and $\lambda_1, \ldots \lambda_k \in \mathbb{R}$, i.e. $\bar{W}^\varrho$ is a centered Gaussian process. Moreover,

$$
\begin{aligned}
\Gamma(A, A') &:= \mathbb{E}[\bar{W}^\varrho(A)\bar{W}^\varrho(A')] \\
&= \mathbb{E}[\{\bar{W}^\varrho(A\setminus(A\cap A')) + \bar{W}^\varrho(A\cap A')\}\{\bar{W}^\varrho(A'\setminus(A\cap A')) + \bar{W}^\varrho(A\cap A')\}] \\
&= \mathbb{E}[\bar{W}^\varrho(A\cap A')^2] \;=\; \varrho(A\cap A').
\end{aligned}
$$

So we obtain with help of Proposition 3.17:

Proposition 5.2 [EXISTENCE OF WHITE NOISE MEASURE] *For every $\varrho(dx) \in \mathcal{M}(\mathbb{R}^m)$, the white noise measure $\bar{W}^\varrho = (\bar{W}^\varrho(A) : A \in \mathcal{A}(\mathbb{R}^m))$ exists and is unique in law. More precisely, $\bar{W}^\varrho$ is a Gaussian process with mean function $\mu \equiv 0$ and covariation function $\Gamma(A, A') = \varrho(A\cap A')$ for $A, A' \in \mathcal{A}(\mathbb{R}^m)$.*

The probability domain of $\bar{W}^\varrho$ is denoted by $[\Omega, \mathcal{F}, \mathbb{P}]$. For many intensity measures $\varrho(dx)$, the samples of the corresponding white noise measure $\bar{W}^\varrho$ will not be σ-additive and so not be (signed) measures. To demonstrate non-σ-additivity for a basic example, let $\bar{W}^\varrho$ be a white noise measure on $\mathcal{A}([0,\infty))$ with intensity measure $\varrho(dt)$ satisfying

$$
\exists \alpha > 0\; \forall T > 0\; \exists c_T > 0 : \quad \sup_{t\leq T} \varrho([0,\infty)\cap B[t,r]) \leq c_T\, r^\alpha \quad \forall r \in (0, T]. \tag{5.1}
$$

Set $w_t^\varrho := \bar{W}^\varrho([0,t])$ for all $t \geq 0$ and note that $w^\varrho = (w_t^\varrho : t \geq 0)$ provides a continuous square-integrable $(\bar{\mathcal{F}}_t)$-martingale with quadratic variation process $\langle w^\varrho\rangle_. = \varrho([0,.])$, where $(\bar{\mathcal{F}}_t)$ denotes the usual augmentation of $(\mathcal{F}_t^{w^\varrho})$. The form of $\langle w^\varrho\rangle$ is justified by

$$
\begin{aligned}
&\mathbb{E}\Big[(w_{t+s}^\varrho)^2 - \varrho([0,t+s]) - \Big((w_t^\varrho)^2 - \varrho([0,t])\Big)\Big|\bar{\mathcal{F}}_t\Big] \\
&= \mathbb{E}\Big[\Big(\bar{W}^\varrho([0,t]) + \bar{W}^\varrho((t,t+s])\Big)^2 - \bar{W}^\varrho([0,t])^2 - \varrho((t,t+s])\Big|\bar{\mathcal{F}}_t\Big] \\
&= \mathbb{E}\Big[2\bar{W}^\varrho([0,t])\bar{W}^\varrho((t,t+s]) + \bar{W}^\varrho((t,t+s])^2 - \varrho((t,t+s])\Big|\bar{\mathcal{F}}_t\Big] \\
&= 2\bar{W}^\varrho([0,t])\mathbb{E}[\bar{W}^\varrho((t,t+s])] + \mathbb{E}[\bar{W}^\varrho((t,t+s])^2] - \varrho((t,t+s]) \\
&= 0 + \varrho((t,t+s]) - \varrho((t,t+s]) \;=\; 0 \qquad \forall s,t \geq 0;
\end{aligned}
$$

recall also Proposition 3.23. The continuity can be assumed by Kolmogorov's criterion (Proposition 3.6) and the following estimate (with $q \geq 1$ sufficiently large, i.e. $q\alpha > 1$):

$$
\mathbb{E}[|w_t^\varrho - w_{t'}^\varrho|^{2q}] = \mathbb{E}[|\bar{W}^\varrho((t,t'])^{2q}] \leq c_{q,T}|t-t'|^{q\alpha} \qquad \forall t \leq t' \leq T.
$$

Note that in the case $\varrho(dt) = dt$, where (5.1) is satisfied with $\alpha = 1$ and $c_T = 2$, w^ϱ is just a Brownian motion in $\mathbb{R}$. Suppose the samples of $\bar{W}^\varrho$ were σ-additive. Then the

[17] Pick a partition $\{A'_i\}_{i=1}^{k'}$ of $\cup_{i=1}^k A_i$ so that each A_i is a (disjoint) union of sets from $\{A'_i\}_{i=1}^{k'}$, and use (i)-(iii) and the fact that a weighted sum of independent Gaussian random variables is again Gaussian.

restriction $\bar{W}^\varrho|_{[0,T]}$ to $[0,T]$ would be a finite signed measure for every $T > 0$. Hence the samples of $\bar{W}^\varrho|_{[0,T]}$ had the decomposition $\bar{W}^\varrho|_{[0,T]} = \bar{W}_+^\varrho|_{[0,T]} - \bar{W}_-^\varrho|_{[0,T]}$ with two finite positive measures $\bar{W}_+^\varrho|_{[0,T]}$ and $\bar{W}_-^\varrho|_{[0,T]}$ (Hahn-Jordan decomposition, cf. [Bau92] Korollar 18.2). We consequently obtained $w_t^\varrho = \bar{W}_+^\varrho|_{[0,T]}([0,t]) - \bar{W}_-^\varrho|_{[0,T]}([0,t])$ for $t \in [0,T]$. That is, w^ϱ had to be the difference of two non-decreasing right-continuous functions and so of bounded variation, i.e.

$$\forall 0 \le s < T: \qquad \sup_{(t_i)_{i=0}^k} \sum_{i=1}^k |w_{t_i}^\varrho - w_{t_{i-1}}^\varrho| < \infty$$

where the supremum ranges over all $(t_i)_{i=0}^k$ with $s = t_0 < \dots < t_k = T$ and $k \ge 1$. But continuous non-constant martingales are of unbounded variation (cf. [RY98], Proposition IV.1.2). This yields a contradiction whence $\bar{W}^\varrho$'s samples cannot be a.s. σ-additive.

5.2 SODEs driven by white noises and Itô's theory

One of the major issues of this thesis is the stochastic heat equation, i.e. a particular stochastic *partial* differential equation (SPDE). The theory of SPDEs belongs to the modern fields of probability theory. A basic work has been published only in 1986 ([Wal86]). On the other hand, the basics of the theory of stochastic *ordinary* differential equations (SODEs) are meanwhile more or less standard (see [IW89], [KS91], [Kal97], [RY98] and many others). A pioneer work appeared already in 1951 ([Itô51]). Since the reader might be more familiar with SODEs, we proceed as follows. In this section (as a first step) we approach Itô's interpretation of certain SODEs by means of white noise measures. In the next section we will see that Walsh's interpretation of certain SPDEs is based on the white noise measure theory in the same way. The SODE we consider in this section is

$$\frac{d}{dt}u_t = b(t,u_t)\frac{\sigma(dt)}{dt}(t) + a(t,u_t)\frac{d}{dt}w_t^\varrho, \qquad t \ge 0 \tag{5.2}$$

where $a, b : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ are continuous functions, $\varrho(dt)$ and $\sigma(dt)$ are Radon measures on $[0,\infty)$ with $\varrho(dt)$ satisfying condition (5.1), $\frac{\sigma(dt)}{dt}$ is the Lebesgue density of $\sigma(dt)$ and $w^\varrho = (w_t^\varrho : t \ge 0)$ is the $\mathcal{M}_c^2$-martingale with quadratic variation process $\langle w^\varrho \rangle_. = \varrho([0,.])$ from Section 5.1. As already mentioned, if $\varrho(dt) = dt$, then w^ϱ is just a Brownian motion in $\mathbb{R}$. In this case, $\frac{d}{dt}w_t^\varrho$ is usually called (standard) white noise. For more general $\varrho(dt)$, $\frac{d}{dt}w_t^\varrho$ is sometimes called *white noise* with intensity measure $\varrho(dt)$. Of course, the formulation of equation (5.2) is rather vague. In Section 5.1 we mentioned the unbounded variation of w^ϱ which indicates an irregular behavior of w^ϱ. Indeed, if $\varrho(dt) \not\equiv 0$, w_t^ϱ will not be everywhere differentiable. A Brownian motion ($\varrho(dt) = dt$) is even nowhere differentiable (cf. Theorem 2.9.18 of [KS91]). Hence $\frac{d}{dt}w_t^\varrho$ and so the r.h.s. of (5.2) fail to be rigorous. Also, we did not require the measure $\sigma(dt)$ to be absolutely continuous w.r.t. the Lebesgue measure. So the density $\frac{\sigma(dt)}{dt}$ might not exist. The way out is to regard the stochastic differential equation (5.2) as a certain stochastic integral equation (SIE). This is Itô's fundamental idea of stochastic calculus. In the remainder of this section we first give a formal justification of this approach, afterwards we state some rigorous definitions and collect a few basic results.

On a very formal level, we integrate both sides of (5.2) against the Lebesgue measure which yields

$$u_t - u_0 = \int_0^t b(r, u_r)\frac{\sigma(dr)}{dr}dr + \int_0^t a(r, u_r)\frac{d}{dr}w_r^\varrho dr \tag{5.3}$$

$$= \int_0^t b(r, u_r)\sigma(dr) + \int_0^t a(r, u_r)dw_r^\varrho. \tag{5.4}$$

By definition we have $w_t^\varrho = \bar{W}^\varrho([0,t])$, $t \geq 0$, where $\bar{W}^\varrho$ is the white noise measure on $\mathcal{A}([0,\infty))$ with intensity measure $\varrho(dt)$. So, still on a formal level, equation (5.4) turns into

$$u_t - u_0 = \int_0^t b(r, u_r)\sigma(dr) + \int_0^t a(r, u_r)\bar{W}^\varrho(dr). \tag{5.5}$$

If the samples of $\bar{W}^\varrho$ were signed measures, then integral equation (5.5) would just involve Lebesgue integrals (in a pathwise sense). However, in the previous section we established non-σ-additivity of $\bar{W}^\varrho$, i.e. $\bar{W}^\varrho$ is merely a finitely additive set function. Thus equation (5.5) cannot be seen rigorously since $\int_0^t X_r\bar{W}^\varrho(dr)$ does not make sense for general progressively measurable integrands X. Now, Itô's basic idea is the following. For simple integrands $X_t = \xi_0\mathbf{1}_{\{0\}}(t) + \sum_{i=0}^\infty \xi_i\mathbf{1}_{(t_i,t_{i+1}]}(t)$ (cf. Section 3.7) $\bar{W}^\varrho$ actually serves (pathwise) as a true measure, i.e. the second integral in (5.5) can be defined by

$$\int_0^t X_r(\omega)\bar{W}^\varrho(\omega, dr) := \sum_{i=0}^n \xi_i(\omega)\bar{W}^\varrho(\omega, (t_i, t_{i+1}] \cap [0,t]), \quad (\omega, t) \in \Omega \times [0,\infty) \tag{5.6}$$

where n is the unique integer for which $t_n \leq t < t_{n+1}$. For more general progressively measurable integrands X, the integral $\int_0^t X_r\bar{W}^\varrho(dr)$ will be defined as a certain L^2-limit of $(\int_0^t X_r^n\bar{W}^\varrho(dr))$ where (X^n) is a sequence of simple integrands approximating X. The reason why some L^2-limit (in contrast to an a.s.-limit) does exist is that the martingales w^ϱ are of bounded quadratic variation (in contrast to unbounded variation), i.e.

$$\forall 0 \leq s < T: \qquad \sup_{(t_i)_{i=0}^k} \sum_{i=1}^k |w_{t_i}^\varrho - w_{t_{i-1}}^\varrho|^2 < \infty$$

where the supremum ranges over all $(t_i)_{i=0}^k$ with $s = t_0 < \cdots < t_k = T$ and $k \geq 1$ (cf. [RY98], Proposition IV.1.18). Before leaving the current heuristic level we would like to recall that w^ϱ can formally be associated with the "distribution function" and $\dot{w}_t^\varrho = \frac{d}{dt}w_t^\varrho$ with the "density" of the white noise measure $\bar{W}^\varrho$. In view of (5.3) and (5.5), the two notions of white noises $\frac{d}{dt}w_t^\varrho$ and $\bar{W}^\varrho$ coincide in "distribution sense". More precisely, the white noise measure $\bar{W}^\varrho$ gives the rigorous meaning to the formal expression $\frac{d}{dt}w_t^\varrho$.

We are now going to give the precise meaning to equation (5.5), and so to SODE (5.2). The first step is the introduction of the Itô-integral. Let $[\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}]$ be some filtered probability space where $(\mathcal{F}_t)$ satisfies the usual conditions. Consider some $(\mathcal{F}_t)$-martingale $M \in \mathcal{M}_c^2$ and let $\mathcal{L}^2(M)$ denote the space of $(\mathcal{F}_t)$-progressively measurable processes $X = (X_t : t \geq 0)$ with $[X]_t^M < \infty$ for all $t > 0$, where

$$[X]_t^M := \Big(\mathbb{E}\Big[\int_0^t X_r^2 d\langle M\rangle_r\Big]\Big)^{1/2}.$$

Then $[.]^M$ imposes a metric structure on $\mathcal{L}^2(M)$ where $[X]^M := \sum_{k=1}^{\infty} 2^{-k}([X]_k \wedge 1)$. For brevity we suppress the superscript M and write $[.]$ instead of $[.]^M$ whenever there is no risk of ambiguity. Let $\mathfrak{S}$ denote the class of all simple processes, cf. Section 3.7. Then (cf. [KS91], p.137):

Proposition 5.3 *$\mathfrak{S}$ is dense in $\mathcal{L}^2(M)$ w.r.t. $[.]$.*

The goal is to define an integral $I_t^M(X) = \int_0^t X_r dM_r$ for $X \in \mathcal{L}^2(M)$. For $X \in \mathfrak{S}$ we set

$$I_t^M(X) := \sum_{i=0}^{n} \xi_i (M_{t_{i+1}\wedge t} - M_{t_i \wedge t}), \qquad t \geq 0 \tag{5.7}$$

when $X_t = \xi_0 \mathbf{1}_{\{0\}}(t) + \sum_{i=0}^{\infty} \xi_i \mathbf{1}_{(t_i, t_{i+1}]}(t)$ and n is the unique integer for which $t_n \leq t < t_{n+1}$. Note that the right hand sides of (5.6) and (5.7) coincide for $M = w^{\varrho}$. It can be shown (cf. Section 3.2B of [KS91]) that the following assertions hold for all $X, X' \in \mathfrak{S}$:

(a) $I^M(X) \in \mathcal{M}_c^2$,

(b) $I^M(\lambda X + \lambda' X') = \lambda I^M(X) + \lambda' I^M(X')$ for all $\lambda, \lambda' \in \mathbb{R}$,

(c) $\mathbb{E}[I_t^M(X)^2] = \mathbb{E}\Big[\int_0^t X_r^2 d\langle M\rangle_r\Big]$ for all $t > 0$ and so $\|I^M(X)\| = [X]$.

Relation (c) is the key for the extension of I^M from $\mathfrak{S}$ to $\mathcal{L}^2(M)$. Let $X \in \mathcal{L}^2(M)$. By Proposition 5.3 there exists a sequence $(X_n) \subset \mathfrak{S}$ with $\lim_{n\to\infty}[X - X_n] = 0$. By means of (b) and (c) we infer that $(I^M(X_n))_{n\geq 1}$ is a Cauchy sequence in the complete metric space $\mathcal{M}_c^2$ w.r.t. $\|.\|$ (recall Proposition 3.21). The limit in $\mathcal{M}_c^2$, denoted by $I^M(X)$, can be shown to be independent of the choice of (X_n) (cf. [KS91] p.139). This justifies

Definition 5.4 [Itô-integral] *Consider $M \in \mathcal{M}_c^2$ and $X \in \mathcal{L}^2(M)$. The Itô-integral of X w.r.t. M is the unique (up to indistinguishability) element $I^M(X) \in \mathcal{M}^2$ which satisfies $\lim_{n\to\infty} \|I^M(X) - I^M(X_n)\| = 0$ for every sequence $(X_n) \subset \mathfrak{S}$ with $\lim_{n\to\infty}[X - X_n] = 0$. We write $I_t^M(X) = \int_0^t X_r dM_r$.*

We obtain the following elementary properties of Itô-integrals (cf. [KS91] p.139/40):

Proposition 5.5 *If $I^M(X)$ is the Itô-integral of $X \in \mathcal{L}^2(M)$ w.r.t. $M \in \mathcal{M}_c^2$, then:*

(i) $I^M(X) \in \mathcal{M}_c^2$,

(ii) $\mathbb{E}[I_t^M(X)^2] = \mathbb{E}[\int_0^t X_r^2 d\langle M\rangle_r]$ $\forall t > 0$ *and so* $\|I^M(X)\| = [X]$,

(iii) $I^M(\lambda X + \lambda' X') = \lambda I^M(X) + \lambda' I^M(X')$ $\forall \lambda, \lambda' \in \mathbb{R}$,

(iv) $\langle I_t^M(X)\rangle = \int_0^t X_r^2 d\langle M\rangle_r$ $\forall t \geq 0$, *$\mathbb{P}$-almost surely.*

In Definition 5.4 we considered only $M \in \mathcal{M}_c^2$. However, the definition of the Itô-integral can be extended to $M \in \mathcal{M}_{c,loc}$ and $X \in \mathcal{L}^2(M)$ where $\mathcal{L}^2(M)$ is defined in the same manner as before. Let (τ_n) denote the sequence of stopping times from the definition of the local martingale M and set $M_t^n := M_{t\wedge\tau_n}$ and $X_t^n := X_{t\wedge\tau_n}$. Then

$$I_t^M(X) := I_t^{M^n}(X^n) \qquad t \in [0,\tau_n], \qquad \forall n \geq 1$$

provides a consistent definition of a continuous local martingale $I^M(X)$ (cf. Definition 3.2.23 of [KS91]). The latter is called Itô-integral of X w.r.t. the local martingale M.

A very basic tool in dealing with stochastic differential equations is the Itô formula which states that a smooth function of a continuous semimartingale is again a continuous semimartingale, and provides its decomposition. A *continuous $(\mathcal{F}_t)$-semimartingale* is defined to be a continuous $(\mathcal{F}_t)$-adapted process X with decomposition $X_t = X_0 + M_t + A_t$ where M is a continuous local martingale and A is the difference $A^+ - A^-$ of two continuous $(\mathcal{F}_t)$-adapted non-decreasing processes A^+, A^- with $A_0^+ = A_0^- = 0$.

Theorem 5.6 [Itô-formula] *Let $f : \mathbb{R} \to \mathbb{R}$ be in $C^2(\mathbb{R})$ and let X be a continuous semimartingale with decomposition $X_t = X_0 + M_t + A_t$. Then, $\mathbb{P}$-almost surely,*

$$f(X_t) = f(X_0) + \int_0^t f'(X_r)dM_r + \int_0^t f'(X_r)dA_r + \frac{1}{2}\int_0^t f''(X_r)d\langle M\rangle_r, \quad t \geq 0.$$

The proof can be found on pages 150-153 of [KS91]. Note that the integrals against dA_r and $d\langle M\rangle_r$ are pathwise defined as Lebesgue-Stieltjes integrals.

We are now in the position to specify solutions of SODE (5.2). Recall that $w^\varrho = (w_t^\varrho : t \geq 0)$ is an $(\mathcal{F}_t)$-martingale in $\mathcal{M}_c^2$ with $\langle w^\varrho\rangle_. = \varrho([0,.])$, where $(\mathcal{F}_t)$ denotes the usual augmentation of $(\mathcal{F}_t^{w^\varrho})$. In view of the formal link of SODE (5.2) to SIE (5.4), a process $u = (u_t : t \geq 0)$ is said to be a *strong solution* to SODE (5.2) with initial condition $\eta \in \mathbb{R}$ if, given $w^\varrho = [w^\varrho, \Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}]$, it is $(\mathcal{F}_t)$-progressively measurable and satisfies

$$u_t = \eta + \int_0^t b(r,u_r)\sigma(dr) + \int_0^t a(r,u_r)dw_r^\varrho \qquad \forall t \geq 0, \quad \mathbb{P}\text{-almost surely.} \tag{5.8}$$

A process u on the domain of any $w^\varrho = [w^\varrho, \Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}]$ is called *weak solution* to SODE (5.2) with initial condition $\eta \in \mathbb{R}$ if u is $(\mathcal{F}_t)$-progressively measurable and (5.8) holds. A solution is said to be *strongly unique* if any two solutions w.r.t. a given w^ϱ are indistinguishable. We say a solution is *weakly unique* if any two solutions coincide in law.

5.3 SPDEs driven by white noises and Walsh's theory

We now turn to one of the major issues of this thesis. Fix $d \geq 1$ and let Δ be the Laplacian in $\mathbb{R}^d$. That is, $\Delta = \sum_{i=1}^d \partial^2/\partial x_i^2$. Consider continuous functions $a, b : [0,\infty)\times\mathbb{R}^d\times\mathbb{R} \to \mathbb{R}$ and $\eta : \mathbb{R}^d \to \mathbb{R}$ as well as Radon measures $\sigma(dtdx)$ and $\varrho(dtdx)$ on $[0,\infty)\times\mathbb{R}^d$. Let $\frac{\sigma(dtdx)}{dtdx}$ be the Lebesgue density of $\sigma(dtdx)$ which might only exist as a generalized function (distribution). Then consider the following stochastic partial differential equation (SPDE):

$$\begin{aligned}
\frac{\partial}{\partial t}u(t,x) &= \frac{1}{2}\Delta u(t,x) \\
&\quad + b(t,x,u(t,x))\frac{\sigma(dtdx)}{dtdx}(t,x) + a(t,x,u(t,x))\frac{\partial^{1+d}}{\partial t\partial x_1\cdots\partial x_d}w^\varrho(t,x) \\
u(0,x) &= \eta(x), \qquad t \geq 0,\ x \in \mathbb{R}^d
\end{aligned} \tag{5.9}$$

where $w^\varrho : \Omega \times [0,\infty) \times \mathbb{R}^d \to \mathbb{R}$ is an inhomogeneous Brownian sheet based on ϱ. The latter is characterized by

$$\bar{W}^\varrho((t,t'] \times (x_1,x_1'] \times \cdots \times (x_d,x_d']) = \Delta_{(t,t'] \times (x_1,x_1'] \times \cdots \times (x_d,x_d']}(w^\varrho) \tag{5.10}$$

where $\bar{W}^\varrho$ is a white noise measure on $\mathcal{A}([0,\infty) \times \mathbb{R}^d)$ with intensity measure $\varrho(dtdx)$ (recall Definition 5.1 and Proposition 5.2) and

$$\Delta_{(t,t'] \times (x_1,x_1'] \times \cdots \times (x_d,x_d']}(w^\varrho) := \sum_v \operatorname{sgn}(v)\ w^\varrho(v).$$

Here the sum ranges over the 2^{1+d} vertices $v = (v_0, v_1, \ldots, v_d)$ of the rectangle $(t,t'] \times (x_1,x_1'] \times \cdots \times (x_d,x_d']$, and $\operatorname{sgn}(v)$ is $+1$ or -1 according as the number of i $(0 \le i \le d)$ satisfying $v_i = x_i$ (where $x_0 := t$) is even or odd. If $d = 1$, then (5.10) equals

$$\bar{W}^\varrho((t,t'] \times (x,x']) = w^\varrho(t',x') - w^\varrho(t',x) - w^\varrho(t,x') + w^\varrho(t,x).$$

In view of (5.10), w^ϱ can formally be associated with the "distribution function" and $\dot{w}^\varrho(t,x) = \frac{\partial^{1+d}}{\partial t \partial x_1 \cdots \partial x_d} w^\varrho(t,x)$ with the "density" of the white noise measure $\bar{W}^\varrho$ (see [Bil95] p.176 for the notion of distribution functions of measures on $\mathbb{R}^n$). If $\varrho(dtdx) = dtdx$, then w^ϱ is just a (standard) Brownian sheet on $[0,\infty) \times \mathbb{R}^d$ and $\dot{w}^\varrho(t,x) = \frac{\partial^{1+d}}{\partial t \partial x_1 \cdots \partial x_d} w^\varrho(t,x)$ is usually called (standard) time-space white noise. For more general $\varrho(dtdx)$ we call $\dot{w}^\varrho(t,x) = \frac{\partial^{1+d}}{\partial t \partial x_1 \cdots \partial x_d} w^\varrho(t,x)$ *time-space white noise* with intensity measure $\varrho(dtdx)$. As in the case of SODE (5.2), the formulation of SPDE (5.9) is rather vague. On the one hand, the measure $\sigma(dtdx)$ is not demanded to be absolutely continuous w.r.t. the Lebesgue measure $dtdx$, i.e. the existence of the density $\frac{\sigma(dtdx)}{dtdx}$ might fail. On the other hand, the time-space version of w^ϱ will not be differentiable in general either (see Chapter 1 of [Wal86] for the case of the standard Brownian sheet). Again, the way out is to regard the stochastic differential equation as a certain stochastic integral equation.

By multiplying both sides of the first equation in (5.9) by $\psi \in C_c^\infty(\mathbb{R}^d)$ and integrating them against the time-space Lebesgue measure we obtain (on a very formal level) as in Section 5.2 (cf. (5.3)-(5.5)):

$$\begin{aligned}
&\langle u(t,.), \psi \rangle - \langle \eta, \psi \rangle \qquad\qquad (5.11)\\
= &\int_0^t \langle \frac{1}{2}\Delta u(r,y), \psi \rangle dr + \int_0^t \int_{\mathbb{R}^d} b(r,y,u(r,y))\psi(y) \frac{\sigma(drdy)}{drdy}(r,y) drdy \\
&\qquad + \int_0^t \int_{\mathbb{R}^d} a(r,y,u(r,y))\psi(y) \frac{\partial^{1+d}}{\partial r \partial y_1 \cdots \partial y_d} w^\varrho(r,y) drdy \\
= &\int_0^t \langle u(r,y), \frac{1}{2}\Delta\psi \rangle dr + \int_0^t \int_{\mathbb{R}^d} b(r,y,u(r,y))\psi(y)\sigma(drdy) \\
&\qquad + \int_0^t \int_{\mathbb{R}^d} a(r,y,u(r,y))\psi(y)\partial^{1+d} w^\varrho(r,y) \\
= &\int_0^t \langle u(r,y), \frac{1}{2}\Delta\psi \rangle dr + \int_0^t \int_{\mathbb{R}^d} b(r,y,u(r,y))\psi(y)\sigma(drdy) \\
&\qquad + \int_0^t \int_{\mathbb{R}^d} a(r,y,u(r,y))\psi(y)\bar{W}^\varrho(drdy).
\end{aligned}$$

Similarly to the case of $\bar{W}^\varrho(dr)$, the samples of $\bar{W}^\varrho(drdy)$ can be shown to be non-σ-additive for plenty of intensity measures $\varrho(dtdx)$ (e.g. for $\varrho(dtdx) = dtdx$). The second integral in (5.11) thus fails to be rigorous in general. But Itô's way out for the case of time white noises (cf. Section 5.2) also works for the case of time-space white noises. For simple integrands $f \in \bar{\mathfrak{S}}$ (cf. Step 2 below) $\bar{W}^\varrho$ again serves as a true measure. So the definition of the stochastic integral $\int\int f(r,y)\bar{W}^\varrho(drdy)$ can be done in a straightforward (pathwise) manner. For more general predictable integrands f (cf. Definition 5.11 below) the stochastic integral will again be understood as a certain L^2-limit of $(\int\int f_n(r,y)\bar{W}^\varrho(drdy))$ where (f_n) is a sequence of simple integrands approximating f. This generalized Itô-approach is due to Walsh ([Wal86]) to whose honor the stochastic integral is called *Walsh-integral*. Note that the white noise measure $\bar{W}^\varrho$ coincides with the white noise $\frac{\partial^{1+d}}{\partial t\partial x_1\cdots\partial x_d}w^\varrho(t,x)$ in "distribution sense". More precisely, $\bar{W}^\varrho$ gives the rigorous meaning to the formal expression $\frac{\partial^{1+d}}{\partial t\partial x_1\cdots\partial x_d}w^\varrho(t,x)$. In the remainder of this section we develop Walsh's theory and end up with a sensible definition of continuous solutions to SPDE (5.9), respectively SIE (5.11). We proceed in four steps.

Step 1 (specification of integrators). The first step is to specify the exact class of integrators. In the case of Itô-integrals the role of integrators was played by certain martingales $M = (M_t : t \geq 0)$. In the present setting integrators are Walsh's so-called martingale measures $M = (M_t(B) : t \geq 0, B \in \mathcal{B}(\mathbb{R}^d))$; cf. [Wal86], [EKM90]. Informally, $M_.(B)$ is a martingale for fixed B, and $M_t(.)$ is a sort of signed measure for fixed t. As our interest is devoted to SPDEs driven by *white* noises, we only focus on so-called *orthogonal* martingale measures. For a more general setting we refer to Chapter 2 of [Wal86]. Let $[\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}]$ be some filtered probability space where $(\mathcal{F}_t)$ satisfies the usual conditions.

Definition 5.7 [σ-FINITE L^2-VALUED MEASURE] *A real-valued process $\Phi = (\Phi(A) : A \in \mathcal{A}(\mathbb{R}^d))$ is called a σ-finite L^2-valued measure on $\mathbb{R}^d$ if the following assertions hold:*

(i) $\sup_{A\in\mathcal{A}([-n,n]^d)} \mathbb{E}[\Phi^2(A)] < \infty$, $\forall n \geq 1$

(ii) $\Phi(A \cup A') = \Phi(A) + \Phi(A')$ $\mathbb{P}$*-almost surely for all disjoint* $A, A' \in \mathcal{A}(\mathbb{R}^d)$

(iii) $\lim_{j\to\infty} \mathbb{E}[\Phi^2(A_j)] = 0$ *for all* $(A_j) \subset \mathcal{A}([-n,n]^d)$ *with* $A_j \downarrow \emptyset$, $\forall n \geq 1$.

It will be extended from $\mathcal{A}(\mathbb{R}^d)$ to $\mathcal{B}(\mathbb{R}^d)$ by setting $\Phi(B) := L^2(\Omega,\mathbb{P})\text{-}\lim_{n\to\infty}\Phi(B\cap[-n,n])$ if the limit exists, and $\Phi(B)$ undefined otherwise, for all $B \in \mathcal{B}(\mathbb{R}^d)$.

Condition *(iii)* is sometimes called L^2-*σ-additivity*, formally justified by the equivalence of σ-additivity and continuity from above of finitely additive set functions, cf. Section 2.1.

Definition 5.8 [ORTHOGONAL MARTINGALE MEASURE] *A real-valued process $M = (M_t(A) : t \geq 0, A \in \mathcal{A}(\mathbb{R}^d))$ is called an orthogonal martingale measure if the following assertions hold:*

(i) $M_t(.)$ *is a σ-finite L^2-valued measure on $\mathbb{R}^d$, $\forall t \geq 0$*

(ii) $(M_t(A) : t \geq 0)$ *is an $(\mathcal{F}_t)$-martingale in $\mathcal{M}^2$, $\forall A \in \mathcal{A}(\mathbb{R}^d)$*

(iii) $(M_t(A) : t \geq 0)$, $(M_t(A') : t \geq 0)$ *are orthogonal,* $\forall$ *disjoint* $A, A' \in \mathcal{A}(\mathbb{R}^d)$.

If the martingales in (ii) are continuous, then M *is said to be continuous.*

The analogue of the quadratic variation process of an $\mathcal{M}^2$-martingale is the quadratic variation measure (cf. Corollary 2.8 of [Wal86]):

Proposition 5.9 [QUADRATIC VARIATION MEASURE] *For every orthogonal martingale measure* M *there is a (unique) random measure* $\langle M\rangle(dtdx)$ *on* $[0,\infty) \times \mathbb{R}^d$ *satisfying:*

(i) $(\langle M\rangle([0,t] \times A) : t \geq 0)$ *is* $(\mathcal{F}_t)$*-predictable* $\forall A \in \mathcal{A}(\mathbb{R}^d)$,

(ii) $\langle M\rangle([0,t] \times A) = \langle M(A)\rangle_t$ $\mathbb{P}$*-almost surely,* $\forall t \geq 0$ *and* $A \in \mathcal{A}(\mathbb{R}^d)$.

The measure $\langle M\rangle(dtdx)$ is called *quadratic variation measure* of M. Note that $M_.(.) = (M_t(.) : t \geq 0)$ can be extended from $\mathcal{A}(\mathbb{R}^d)$ to those $B \in \mathcal{B}(\mathbb{R}^d)$ for which we have $\mathbb{E}[\langle M\rangle([0,t] \times B)] < \infty$ $\forall t \geq 0$. More precisely, for each such B we can define $M_.(B)$ as the $\|.\|$-limit of $M_.(B \cap [-n,n]^d)$ as $n \to \infty$. A fundamental example for an orthogonal martingale measure is induced by the white noise measure on $\mathcal{A}([0,\infty) \times \mathbb{R}^d)$:

Example 5.10 [WHITE NOISE] *Let* $\bar{W}^\varrho$ *be a white noise measure on* $\mathcal{A}([0,\infty) \times \mathbb{R}^d)$ *with intensity measure* $\varrho(dtdx) \in \mathcal{M}([0,\infty) \times \mathbb{R}^d)$. *Suppose* $\varrho(dtdx) = \varrho_1(t,dx)\varrho_2(dt)$, *where* ϱ_1 *is a kernel from* $\mathbb{R}^d$ *to* $\mathbb{R}$ *and* $\varrho_2(dt) \in \mathcal{M}([0,\infty))$. *Further assume* $\sup_{t \leq T} \varrho_1(t,A) < \infty$, *for all* $T > 0$ *and* $A \in \mathcal{A}(\mathbb{R}^d)$, *as well as:*

$$\exists \alpha_2 > 0 \, \forall T > 0 \, \exists c_T > 0 : \quad \sup_{t \leq T} \varrho_2([0,\infty) \cap B[t,r]) \leq c_T \, r^{\alpha_2} \quad \forall r \in (0,1]. \tag{5.12}$$

Set $W_t^\varrho(A) := \bar{W}^\varrho([0,t] \times A)$ *for all* $t \geq 0$ *and* $A \in \mathcal{A}(\mathbb{R}^d)$, *and let* $(\bar{\mathcal{F}}_t)$ *denote the usual augmentation of* $(\mathcal{F}_t)$ *where* $\mathcal{F}_t := \sigma(W_s^\varrho(A) : s \leq t, A \in \mathcal{A}(\mathbb{R}^m))$ *for every* $t \geq 0$. *Then* $W^\varrho = (W_t^\varrho(A) : t \geq 0, A \in \mathcal{A}(\mathbb{R}^d))$ *is a continuous orthogonal martingale measure w.r.t.* $(\bar{\mathcal{F}}_t)$ *and has (deterministic) quadratic variation measure* $\langle W^\varrho\rangle(dtdx) = \varrho(dtdx)$.

Proof First of all we note that any $(W_t^\varrho(A) : t \geq 0)$ may be assumed to be continuous. Indeed, by (5.12) we obtain for $m \geq 1$ and $A \in \mathcal{A}(\mathbb{R}^d)$:

$$\mathbb{E}[|W_t^\varrho(A) - W_{t'}^\varrho(A)|^{2m}] = \mathbb{E}[|\bar{W}^\varrho((t,t'] \times A)^{2m}] \leq c_{m,T,A}|t-t'|^{m\alpha_2}$$

for all $t, t' \in [0,T]$ with $|t-t'| \leq 1$. For m sufficiently large (i.e. $m\alpha_2 > 1$) we infer by means of Kolmogorov's continuity criterion (Proposition 3.6) the existence of a continuous modification $\tilde{W}_.^\varrho(A)$ of $W_.^\varrho(A)$. Then $\tilde{W}^\varrho := (\tilde{W}_t^\varrho(A) : t \geq 0, A \in \mathcal{A}(\mathbb{R}^d))$ is a continuous orthogonal martingale measure (w.r.t. $(\bar{\mathcal{F}}_t)$, recall Proposition 3.23) with $\langle\tilde{W}^\varrho\rangle(dtdx) = \varrho(dtdx)$. The key for the proof of conditions $(i)-(iii)$ of Definition 5.8 and the form of $\langle W^\varrho\rangle(dtdx)$ is the independence of $\bar{W}^\varrho(C)$ and $\bar{W}^\varrho(C')$ for disjoint $C, C' \in \mathcal{A}([0,\infty) \times \mathbb{R}^d)$, i.e. (ii) of Definition 5.1. We omit the details. The continuity is obvious. Be aware that the process $W^\varrho = (W_t^\varrho(A) : t \geq 0, A \in \mathcal{A}(\mathbb{R}^d))$ is, in fact, meant to be the process $\tilde{W}^\varrho$. $\square$

Step 2 (specification of integrands). In the second step we specify admissible integrands for a given integrator. A function $f : \Omega \times [0, \infty) \times \mathbb{R}^d \to \mathbb{R}$ is said to be *simple* if it is of the form

$$f(\omega, t, x) = \xi_0(\omega)\mathbf{1}_{\{0\}}(t)\mathbf{1}_{B_0}(x) + \sum_{i=0}^{\infty} \xi_i(\omega)\mathbf{1}_{(t_i,t_{i+1}]}(t)\mathbf{1}_{B_i}(x)$$

where $0 = t_0 < t_1 < \cdots < t_i < \cdots \to \infty$, $B_i \in \mathcal{B}(\mathbb{R}^d)$, $\exists c > 0 : \sup_{i\geq 0} \xi_i(\omega) < c \ \forall\omega$ and ξ_i is $\mathcal{F}_{t_i}$-measurable. We write $\bar{\mathfrak{S}}$ for the class of simple functions and denote the σ-algebra in $\Omega \times [0, \infty) \times \mathbb{R}^d$ generated by $\bar{\mathfrak{S}}$ by $\bar{\mathcal{F}}_{pred}$. An $\bar{\mathcal{F}}_{pred}$-measurable function $f : \Omega \times [0, \infty) \times \mathbb{R}^d \to \mathbb{R}$ is called $(\mathcal{F}_t)$-*predictable.* Now, fix an orthogonal martingale measure M and let $\mathcal{P}^2(M)$ be the space of predictable functions f satisfying $|[f]|_t^M < \infty$ for all $t > 0$ where

$$|[f]|_t^M := \Big(\mathbb{E}\Big[\int_0^t \int_{\mathbb{R}^d} f^2(r, y)\langle M\rangle(drdy)\Big]\Big)^{1/2}.$$

Definition 5.11 *The space $\mathcal{P}^2(M)$ is called the class of admissible integrands w.r.t. M.*

If there is no risk of ambiguity, then we suppress the superscript M and write $|[.]|$ instead of $|[.]|^M$. $|[.]|$ imposes a metric structure[18] on $\mathcal{P}^2(M)$ where $|[f]| := \sum_{k=1}^{\infty} 2^{-k}(|[f]|_k \wedge 1)$. Note that $\mathcal{P}^2(M)$ is complete w.r.t. $|[.]|$ and we have (cf. [Wal86] 2.5, resp. 2.3):

Proposition 5.12 *$\bar{\mathfrak{S}}$ is dense in $\mathcal{P}^2(M)$ w.r.t. $|[.]|$.*

Step 3 (construction of the integral). Our intension here is to define the integral $f \cdot M_t(A) = \int_0^t \int_A f(r, y)M(drdy)$ for $f \in \mathcal{P}^2(M)$. For $f \in \bar{\mathfrak{S}}$ we set[19]

$$f \cdot M_t(A) := \sum_{i=0}^{n} \xi_i[M_{t_{i+1}\wedge t}(B_i \cap A) - M_{t_i\wedge t}(B_i \cap A)], \qquad t \geq 0, A \in \mathcal{A}(\mathbb{R}^d) \tag{5.13}$$

when $f(t, x) = \xi_0\mathbf{1}_{\{0\}}(t)\mathbf{1}_{B_0}(x) + \sum_{i=0}^{\infty} \xi_i\mathbf{1}_{(t_i,t_{i+1}]}(t)\mathbf{1}_{B_i}(x)$ and n is the unique integer for which $t_n \leq t < t_{n+1}$. The process $f \cdot M$ is again an orthogonal martingale measure. For $f, f' \in \bar{\mathfrak{S}}$ we further obtain (see also [Wal86], Exercise 2.6):

(a) $f \cdot M$ is continuous if M is,

(b) $(\lambda f + \lambda' f') \cdot M = \lambda(f \cdot M) + \lambda'(f' \cdot M)$ for all $\lambda, \lambda' \in \mathbb{R}$,

(c) $\mathbb{E}[(f \cdot M_t(A))^2] \leq \mathbb{E}[\int_0^t \int_{\mathbb{R}^d} f^2(r, y)\langle M\rangle(drdy)]$ for all $t > 0$ and $A \in \mathcal{B}(\mathbb{R}^d)$.

Recall the definition of $\|.\|_t$ and $\|.\|$ from (3.13). Then (c) immediately yields

(d) $\|f \cdot M(A)\|_t \leq |[f]|_t \ \forall t > 0, A \in \mathcal{A}(\mathbb{R}^d)$, and so $\|f \cdot M(A)\| \leq |[f]| \ \forall A \in \mathcal{A}(\mathbb{R}^d)$.

[18]In fact, $d_{|[.]|}(f, f') := |[f - f']|$ provides a metric on $\mathcal{P}^2(M)$.

[19]If M is W^ϱ from Example 5.10, then (5.13) equals $f \cdot W_t^\varrho(A) := \sum_{i=0}^{n} \xi_i \bar{W}^\varrho(((t_i, t_{i+1}] \cap [0, t]) \times (B_i \cap A))$.

Now, let $f \in \mathcal{P}^2(M)$. According to Proposition 5.12 there exists a sequence $(f_n) \subset \bar{\mathfrak{S}}$ with $\lim_{n\to\infty} |[f - f_n]| = 0$. By means of (b) and (d) we infer that $(f_n \cdot M(A))_{n\geq 1}$ is a Cauchy sequence in the complete metric space $(\mathcal{M}^2, \|.\|)$ (recall Proposition 3.21). The limit in $\mathcal{M}^2$ is denoted by $f \cdot M(A)$ and independent of the choice of (X_n) (cf. [Wal86], p.295). We get the following essential properties (cf. [Wal86], Theorem 2.5):

Proposition 5.13 *For $f, f' \in \mathcal{P}^2(M)$ we have:*

(i) $f \cdot M = (f \cdot M_t(A) : t \geq 0, A \in \mathcal{A}(\mathbb{R}^d))$ *is an orthogonal martingale measure,*

(ii) $\mathbb{E}[(f \cdot M_t(A))^2] \leq \mathbb{E}[\int_0^t \int_{\mathbb{R}^d} f^2(r,y)\langle M\rangle(drdy)]\ \forall t > 0, A \in \mathcal{A}(\mathbb{R}^d)$,

(iii) $\|f \cdot M(A)\|_t \leq |[f]|_t\ \forall t > 0, A \in \mathcal{A}(\mathbb{R}^d)$,

(iv) $(\lambda f + \lambda' f') \cdot M = \lambda(f \cdot M) + \lambda'(f' \cdot M)\ \forall \lambda, \lambda' \in \mathbb{R}$,

(v) $\langle f \cdot M\rangle(dtdx) = f^2(t,x)\langle M\rangle(dtdx)$.

Assertion (i) and the above considerations justify the following definition.

Definition 5.14 [Walsh-integral] *Consider an orthogonal martingale measure M and $f \in \mathcal{P}^2(M)$. Let $f \cdot M(A)$ be the unique (up to indistinguishability) element of $\mathcal{M}^2$ for which $\lim_{n\to\infty} \|f \cdot M(A) - f_n \cdot M(A)\| = 0$ for every sequence $(f_n) \subset \bar{\mathfrak{S}}$ with $\lim_{n\to\infty} |[f - f_n]| = 0$, $\forall A \in \mathcal{A}(\mathbb{R}^d)$. Then the orthogonal martingale measure $f \cdot M = (f \cdot M_t(A) : t \geq 0, A \in \mathcal{A}(\mathbb{R}^d))$ is called Walsh-integral of f w.r.t. M. We write $f \cdot M_t(A) = \int_0^t \int_A f(r,y)M(drdy)$.*

Remark 5.15 [continuity] *If the orthogonal martingale measure M is continuous, then the Walsh-integral $f \cdot M$ of f w.r.t. M is continuous either. In fact, for continuous M one can take the limit of $(f_n \cdot M(A))$ in $(\mathcal{M}_c^2, \|.\|)$ rather than in $(\mathcal{M}^2, \|.\|)$.*

Proposition 5.16 [chain rule] *Let M be an orthogonal martingale measure and $f \in \mathcal{P}^2(M)$. Then we have $gf \in \mathcal{P}^2(M)$ and $g \cdot (f \cdot M) = (gf) \cdot M$ for every $g \in \mathcal{P}^2(f \cdot M)$.*

Proof Since $g \in \mathcal{P}^2(f \cdot M)$, $gf \in \mathcal{P}^2(M)$ follows from Proposition 5.13 (v). To prove the chain rule we first consider the case of elementary simple functions

$$f(\omega,t,x) = \mathbf{1}_{(u_1,v_1]}(t)\mathbf{1}_{B_1}(x)\xi_1(\omega) \quad \text{and} \quad g(\omega,t,x) = \mathbf{1}_{(u_2,v_2]}(t)\mathbf{1}_{B_2}(x)\xi_2(\omega).$$

Here the claim is a consequence of

$$\begin{aligned}
(g \cdot (f \cdot M))_t(A) &= ((\mathbf{1}_{(u_2,v_2]}\mathbf{1}_{B_2}\xi_2) \cdot (f \cdot M))_t(A) \\
&= \xi_2[(f \cdot M)_{t\wedge v_2}(A \cap B_2) - (f \cdot M)_{t\wedge u_2}(A \cap B_2)] \\
&= \xi_2[(\mathbf{1}_{(u_1,v_1]}\mathbf{1}_{B_1}\xi_1 \cdot M)_{t\wedge v_2}(A \cap B_2) - (\mathbf{1}_{(u_1,v_1]}\mathbf{1}_{B_1}\xi_1 \cdot M)_{t\wedge u_2}(A \cap B_2)] \\
&= \xi_2[\xi_1[M_{(t\wedge v_2)\wedge v_1}((A \cap B_2) \cap B_1) - M_{(t\wedge v_2)\wedge u_1}((A \cap B_2) \cap B_1)] \\
&\qquad -\xi_1[M_{(t\wedge u_2)\wedge v_1}((A \cap B_2) \cap B_1)] - M_{(t\wedge u_2)\wedge u_1}((A \cap B_2) \cap B_1)]] \\
&= (\xi_2\xi_1[\mathbf{1}_{(v_2\wedge u_1, v_2\wedge v_1]} - \mathbf{1}_{(u_2\wedge u_1, u_2\wedge v_1]}]\mathbf{1}_{B_1\cap B_2}) \cdot M_t(A) \\
&= ((\xi_2\mathbf{1}_{(u_2,v_2]}\mathbf{1}_{B_2})(\xi_1\mathbf{1}_{(u_1,v_1]}\mathbf{1}_{B_1})) \cdot M_t(A) \;=\; (gf) \cdot M_t(A) \quad \forall t \geq 0, A \in \mathcal{A}(\mathbb{R}^d).
\end{aligned}$$

The stepwise extension of the claim to general $f, g \in \bar{\mathfrak{S}}$ and general $f \in \mathcal{P}^2(M), g \in \mathcal{P}^2(f \cdot M)$ follows by linearity and a proper approximation. We omit the details. □

Lemma 5.17 [ORTHOGONALITY] *Let M and N be two orthogonal martingale measures, $f \in \mathcal{P}^2(M)$ and $g \in \mathcal{P}^2(N)$. If $M(A)$ and $N(A')$ are orthogonal martingales for every $A, A' \in \mathcal{A}(\mathbb{R}^d)$, then the same is true for $f \cdot M(A)$ and $g \cdot N(A')$.*

Of course, if M and N are independent, then the martingales $M(A)$ and $N(A')$ are orthogonal for every $A, A' \in \mathcal{A}(\mathbb{R}^d)$, and the lemma implies that the same is true for the martingales $f \cdot M(A)$ and $g \cdot N(A')$. This fact will be needed later on.

Proof First consider elementary simple functions $f = \xi \mathbf{1}_{(u_1,u_2]} \mathbf{1}_B$ and $g = \eta \mathbf{1}_{(v_1,v_2]} \mathbf{1}_C$. In this case the corresponding Walsh-integrals can be expressed in terms of Itô-integrals. Indeed, if we set $\tilde{f} := \xi \mathbf{1}_{(u_1,u_2]}$ and $\tilde{g} := \eta \mathbf{1}_{(v_1,v_2]}$, then we have $f \cdot M_t(A) = I_t^{M(A\cap B)}(\tilde{f})$ and $g \cdot N_t(A') = I_t^{N(A'\cap C)}(\tilde{g})$ for every $A, A' \in \mathcal{A}(\mathbb{R}^d)$. So the claim of the lemma follows immediately from the cross-variation formula for Itô-integrals (cf. [KS91], p.142):

$$\langle I^{M^1}(X^1), I^{M^2}(X^2) \rangle_t = \int_0^t X_r^1 X_r^2 \, d\langle M^1, M^2 \rangle_r \qquad \forall t \geq 0, \quad \mathbb{P}\text{-almost surely.}$$

For general simple functions $f, g \in \bar{\mathfrak{S}}$ the claim of the lemma can be inferred by means of Lemma 3.27. Hence we know that the process $(f \cdot M(A))(g \cdot N(A'))$ is a martingale for every $f, g \in \bar{\mathfrak{S}}$ and $A, A' \in \mathcal{A}(\mathbb{R}^d)$.

Now, pick arbitrary $f \in \mathcal{P}^2(M)$ and $g \in \mathcal{P}^2(N)$. By Proposition 5.12 the functions f and g can be approximated w.r.t. $|[.]|^M$, respectively $|[.]|^N$, by sequences (f_n) and (g_n) of simple functions. Moreover, by Hölder's inequality and Proposition 5.13(iii), we have for every $t \geq 0$ and $A, A' \in \mathcal{A}(\mathbb{R}^d)$:

$$\begin{aligned}
&\|(f \cdot M(A))(g \cdot N(A')) - (f_n \cdot M(A))(g_n \cdot N(A'))\|_t \\
&= \mathbb{E}\Big[\Big|(f \cdot M_t(A))(g \cdot N_t(A')) - (f_n \cdot M_t(A))(g_n \cdot N_t(A'))\Big|^2\Big]^{1/2} \\
&\leq c\Big(\mathbb{E}\Big[\Big|(f \cdot M_t(A))\Big|^2\Big]^{1/2} \mathbb{E}\Big[\Big|g \cdot N_t(A') - g_n \cdot N_t(A')\Big|^2\Big]^{1/2}\Big)^{1/2} \\
&\quad + \Big(\mathbb{E}\Big[\Big|(g_n \cdot N_t(A'))\Big|^2\Big]^{1/2} \mathbb{E}\Big[\Big|f \cdot M_t(A) - f_n \cdot M_t(A)\Big|^2\Big]^{1/2}\Big)^{1/2} \\
&\leq c\Big(\|g \cdot N(A') - g_n \cdot N(A')\|_t + \|f \cdot M(A) - f_n \cdot M(A)\|_t\Big)^{1/2} \\
&\leq c\Big(|[g - g_n]|_t^N + |[f - f_n]|_t^M\Big)^{1/2}.
\end{aligned}$$

The latter estimate converges to zero as $n \to \infty$. Hence, $(f \cdot M(A))(g \cdot N(A'))$ is the $\|.\|$-limit of the $\mathcal{M}^2$-martingales $(f_n \cdot M(A))(g_n \cdot N(A'))$. Therefore, $(f \cdot M(A))(g \cdot N(A'))$ is a martingale, i.e. $f \cdot M(A)$ and $g \cdot N(A')$ are orthogonal. □

Lemma 5.18 [Walsh- and Itô-integral related] *Let M be an orthogonal martingale measure and $f \in \mathcal{P}^2(M)$ be independent of x, i.e. $f(\omega,t,x) = f(\omega,t)$. Then $f \in \mathcal{L}^2(M(A))$ and $f \cdot M_t(A) = I_t^{M(A)}(f)$ $\forall t \geq 0$ $\mathbb{P}$-almost surely, for every $A \in \mathcal{A}(\mathbb{R}^d)$.*

Proof Since f is $\bar{\mathcal{F}}_{pred}$-measurable and independent of x, it can also be seen as an $\mathcal{F}_{pred}$-measurable function. Further, by Proposition 5.9(ii) we obtain for all $t \geq 0$:

$$\begin{aligned} [f]_t^2 &= \mathbb{E}\Big[\int_0^t f^2(r) d\langle M(A)\rangle_r\Big] = \mathbb{E}\Big[\int_0^t f^2(r) d\langle M\rangle([0,.] \times A)_r\Big] \\ &= \mathbb{E}\Big[\int_0^t \int_A f^2(r)\langle M\rangle(drdy)\Big] \leq \mathbb{E}\Big[\int_0^t \int_{\mathbb{R}^d} f^2(r)\langle M\rangle(drdy)\Big] = |[f]|_t^2 < \infty. \end{aligned} \tag{5.14}$$

Hence, $f \in \mathcal{L}^2(M(A))$. By Proposition 5.12, f can be approximated w.r.t. $|[.]|$ by simple functions $f_n = \xi_0^n \mathbf{1}_{\{0\}} + \sum_{i=0}^\infty \xi_i^n \mathbf{1}_{(t_i^n, t_{i+1}^n]}$. In view of (5.14), f_n also approximates f w.r.t. $[.]$. Therefore, we obtain with help of Propositions 5.5(ii) and 5.13(iii):

$$\begin{aligned} f \cdot M_.(A) &= \int_0^. \int_A f(r) M(drdy) = \lim_{n\to\infty} \int_0^. \int_A f_n(r) M(drdy) \\ &= \lim_{n\to\infty} \sum_{i: t_i^n \leq (.)} \xi_i^n \Big[M_{t_{i+1}^n \wedge .}(A) - M_{t_i^n \wedge .}(A)\Big] \\ &= \lim_{n\to\infty} \int_0^. f_n(r) dM_r(A) = \int_0^. f(r) dM_r(A) = I_.^{M(A)}(f) \end{aligned}$$

where the convergence holds w.r.t. $\|.\|$. This proves the claim. □

Note that the statements of Lemmas 5.17 and 5.18 remain true for those $B \in \mathcal{B}(\mathbb{R}^d)$ (instead of $A \in \mathcal{A}(\mathbb{R}^d)$) which the involved integrals can be extended to. The following result provides a kind of Fubini theorem (cf. Theorem 2.6 of [Wal86]).

Proposition 5.19 [stochastic Fubini] *Let M be an orthogonal martingale measure and consider some finite measure space $[S, \mathcal{S}, \mu]$. Suppose $f : [0,\infty) \times \mathbb{R} \times \Omega \times S \to \mathbb{R}$ is $\bar{\mathcal{F}}_{pred} \times \mathcal{S}$-measurable and satisfies*

$$\mathbb{E}\Big[\int_S \int_0^t \int_{\mathbb{R}} |f(r,y,s)|^2 \langle M\rangle(drdy)\mu(ds)\Big] < \infty \qquad \forall t > 0.$$

Then we have $\mathbb{P}$-almost surely

$$\int_S \int_0^t \int_{\mathbb{R}} f(r,y,s) M(drdy)\mu(ds) = \int_0^t \int_{\mathbb{R}} \int_S f(r,y,s)\mu(ds) M(drdy), \qquad t \geq 0.$$

Step 4 (specification of SPDE (5.9)). We are now in the position to specify solutions to SPDE (5.9). The state space $\mathbb{F}$ of a solution $u = (u(t,.) : t \geq 0)$ will always be one of the spaces $C_{tem}(\mathbb{R}^d)$, $C_{rap}(\mathbb{R}^d)$ or $C_b(\mathbb{R}^d)$. While $C_{rap}(\mathbb{R}^d)$ will be furnished with the metric d_{rap}, we equip $C_{tem}(\mathbb{R}^d)$ and $C_b(\mathbb{R}^d)$ with the metric d_{tem}. Recall that

$a, b : [0, \infty) \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ are continuous functions and $\sigma(dtdx)$ and $\varrho(dtdx)$ are Radon measures on $[0, \infty) \times \mathbb{R}^d$. Let $W^\varrho = [(W^\varrho_t(A) : t \geq 0, A \in \mathcal{A}(\mathbb{R}^d)), \Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}]$ be a continuous orthogonal martingale measure with quadratic variation measure $\varrho(dtdx)$; cf. Example 5.10. Note that we required $(\mathcal{F}_t)$ to satisfy the usual conditions. In view of the formal link of SPDE (5.9) to SIE (5.11) we define:

Definition 5.20 [SOLUTIONS, SPDE] *(i) Given the continuous orthogonal martingale measure $W^\varrho = [W^\varrho, \Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}]$, a real-valued $(\mathcal{F}_t)$-predictable process $u = (u(t, x) : t \geq 0, x \in \mathbb{R}^d)$ on $[\Omega, \mathcal{F}, \mathbb{P}]$ is said to be a strong $\mathbb{F}$-valued solution to SPDE (5.9) with initial condition $\eta \in \mathbb{F}$ if $(u(t, .) : t \geq 0)$ is $\mathbb{F}$-valued continuous and*

$$\begin{aligned} \langle u(t, .), \psi \rangle &= \langle \eta, \psi \rangle + \int_0^t \langle u(r, .), \frac{1}{2} \Delta \psi \rangle dr \\ &+ \int_0^t \int_{\mathbb{R}^d} b(r, y, u(r, y)) \psi(y) \sigma(drdy) + \int_0^t \int_{\mathbb{R}^d} a(r, y, u(r, y)) \psi(y) W^\varrho(drdy) \end{aligned} \tag{5.15}$$

holds for all $t \geq 0$ and $\psi \in C_c^\infty(\mathbb{R}^d)$, $\mathbb{P}$-almost surely. If one can find any continuous orthogonal martingale measure $W^\varrho = [W^\varrho, \Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}]$ (with quadratic variation measure $\varrho(dtdx)$) and a real-valued $(\mathcal{F}_t)$-predictable process $u = (u(t, x) : t \geq 0, x \in \mathbb{R}^d)$ on $[\Omega, \mathcal{F}, \mathbb{P}]$ such that $(u(t, .) : t \geq 0)$ is $\mathbb{F}$-valued continuous and (5.15) holds, then u is called weak solution to SPDE (5.9) with initial condition $\eta \in \mathbb{F}$.

(ii) An $\mathbb{F}$-valued solution to SPDE (5.9) with initial condition $\eta \in \mathbb{F}$ is called strongly unique if any two solutions w.r.t. a given martingale measure $W^\varrho = [W^\varrho, \Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}]$ are indistinguishable. It is said to be weakly unique if any two solutions (which might be defined on different probability spaces) induce the same law on $[C([0, \infty), C_{tem}(\mathbb{R}^d)), \mathcal{B}_{tem,\infty}]$.[20]

If the noise coefficient a or the quadratic variation measure $\varrho(dtdx)$ of W^ϱ are trivial, i.e. if $a \equiv 0$ or $\varrho(dtdx) \equiv 0$, then SPDE (5.9) turns into the deterministic PDE

$$\begin{aligned} \frac{\partial}{\partial t} u(t, x) &= \frac{1}{2} \Delta u(t, x) + b(t, x, u(t, x)) \frac{\sigma(dtdx)}{dtdx}(t, x) \\ u(0, x) &= \eta(x) \qquad t \geq 0,\ x \in \mathbb{R}^d. \end{aligned} \tag{5.16}$$

Again, the Lebesgue density $\frac{\sigma(dtdx)}{dtdx}$ of $\sigma(dtdx)$ might fail to exist since $\sigma(dtdx)$ is not required to be absolutely continuous w.r.t. the time-space Lebesgue measure. So we once more regard a differential equation as an integral equation:

Definition 5.21 [SOLUTIONS, PDE] *A deterministic $\mathbb{F}$-valued continuous process $(u(t, .) : t \geq 0)$ is said to be an $\mathbb{F}$-valued solution to PDE (5.16) with initial condition $\eta \in \mathbb{F}$ if*

$$\langle u(t, .), \psi \rangle = \langle \eta, \psi \rangle + \int_0^t \langle u(r, .), \frac{1}{2} \Delta \psi \rangle dr + \int_0^t \int_{\mathbb{R}^d} b(r, y, u(r, y)) \psi(y) \sigma(drdy) \tag{5.17}$$

holds for all $t \geq 0$ and $\psi \in C_c^\infty(\mathbb{R}^d)$. An $\mathbb{F}$-valued solution to PDE (5.16) with initial condition $\eta \in \mathbb{F}$ is said to be unique if any two solutions coincide pointwise.

[20] $\mathcal{B}_{tem,\infty}$ is the Borel σ-algebra w.r.r. $d_{tem,\infty}$. See also the footnote on p.105.

5.4 Some remarks

In contrast to PDE (5.16), SPDE (5.9) will only be considered for space dimension $d = 1$. This restriction is not just a convention but it corresponds to a real obstruction for $d > 1$. Heuristic arguments indicate that, if SPDE (5.9) possesses jointly continuous solutions in the sense of Definition 5.20, then the condition

$$\int_0^t \int_{\mathbb{R}^d} p_{t-r}(x,y)\sigma(drdy) + \int_0^t \int_{\mathbb{R}^d} p_{t-r}^2(x,y)\varrho(drdy) < \infty \qquad \forall t > 0, x \in \mathbb{R}^d \tag{5.18}$$

should be fulfilled. And for $d \geq 2$ there does not exist any measure $\varrho(dtdx)$ which might satisfy (5.18). We can also give an example where a solution exists in dimension one but fails to exist for $d \geq 2$. Let $w = w^\varrho$ be a classical Brownian sheet, i.e. $\varrho(dtdx) = dtdx$, and consider the equation

$$\frac{\partial}{\partial t}u(t,x) = \frac{1}{2}\Delta u(t,x) + \sqrt{u(t,x)}\frac{\partial^{1+d}}{\partial t \partial x_1 \cdots \partial x_d}w(t,x), \qquad t \geq 0, x \in \mathbb{R}^d. \tag{5.19}$$

If $d = 1$, then SPDE (5.19) has a continuous $C_{rap}(\mathbb{R}^1)$-valued solution, cf. [MP92], [Shi94]. For $d \geq 2$, however, there is no such solution. Indeed, suppose there was a continuous $C_{rap}^+(\mathbb{R}^d)$-valued solution for $d \geq 2$. Then this solution, regarded as the density process of an $\mathcal{M}_f(\mathbb{R}^d)$-valued process, was the unique solution to the martingale problem for the classical super-Brownian motion (SBM), cf. Section 9.7. However, in [DH79] the states of the classical SBM in $\mathbb{R}^d$, $d \geq 2$, were shown to be singular w.r.t. the Lebesgue measure. This gives a contradiction. Consequently there is no $C_{rap}^+(\mathbb{R}^d)$-valued solution for $d \geq 2$.

If $d = 1$ and $\varrho(dtdx) = \sigma(dtdx) = dtdx$, as a result of which condition (5.18) is fulfilled, then not only SPDE (5.19) but also SPDE (5.9) has jointly continuous solutions, provided the coefficients a and b are sufficiently regular ([Iwa87], [MP92], [Shi94]). Shiga ([Shi94]) worked with the weakest assumptions on a and b. In Chapter 6 we prove that one still obtains jointly continuous solutions when considering more general (possibly singular) measures $\varrho(dtdx)$ and $\sigma(dtdx)$. In fact, we may consider measures $\varrho(dtdx)$ and $\sigma(dtdx)$ that satisfy condition (A), respectively (B). Conditions (A) and (B) were defined in Definitions 2.21 and 2.22, respectively. They are fixed in such a way that they select (nearly all) measures $\varrho(dtdx)$ and $\sigma(dtdx)$ satisfying (5.18). PDE (5.16) can be shown to possess jointly continuous solutions even in higher dimensions $d \geq 2$ (see Chapter 7). Note that condition (B) and, in particular, condition (5.18) (with $\varrho(dtdx) \equiv 0$) can be fulfilled in any dimension $d \geq 1$; examples were given in Section 2.8.

As discussed above, SPDE (5.19) does not possess solutions in the sense of Definition 5.20 for $d \geq 2$. However, we can assign solutions to SPDE (5.19) also in higher dimensions, but in an even weaker sense than the one of Definition 5.20. In fact: According to Theorem 9.21 below, the catalytic SBM $\bar{X} = (\bar{X}_t(dx) : t \geq 0)$ with catalyst $\varrho(dtdx)$ subject to condition (B) satisfies

$$\langle \bar{X}_t, \psi \rangle = \langle \bar{X}_0, \psi \rangle + \int_0^t \langle \bar{X}_r, \frac{1}{2}\Delta\psi \rangle dr + \int_0^t \int_{\mathbb{R}^d} \psi(y) M(drdy), \;\; t \geq 0, \psi \in C_c^\infty(\mathbb{R}^d) \tag{5.20}$$

where M is an orthogonal martingale measure with $\langle M \rangle(dtdx) = C_{[\bar{X},\varrho]}(dtdx)$ (:= collision measure of $\bar{X}$ and ϱ which is defined in Section 9.6). If $\bar{X}$ had a continuous density field

X (which cannot be the case for $d \geq 2$), then we obtained $M(dtdx) = \sqrt{X_t(x)} W^{\varrho}(dtdx)$. To some extent this justifies calling $\bar{X}$ an $\mathcal{M}_f(\mathbb{R}^d)$-*valued solution* of

$$\frac{\partial}{\partial t} X_t(x) = \frac{1}{2}\Delta X_t(x) + \sqrt{X_t(x)} \frac{\partial^{1+d}}{\partial t \partial x_1 \cdots \partial x_d} w^{\varrho}(t,x), \qquad t \geq 0, x \in \mathbb{R}^d \tag{5.21}$$

where w^{ϱ} is an inhomogeneous Brownian sheet based on $\varrho(dtdx)$. For $d \geq 2$ and more general coefficients a and b (than $a(t,x,u) = \sqrt{u}$ and $b \equiv 0$), there seems to be no general statement on SPDE (5.9) in literature so far. It even looks rather difficult to formulate a definition of ($\mathcal{M}_f(\mathbb{R}^d)$-valued) solutions.

Now one could ask whether the state space $\mathcal{M}_f(\mathbb{R}^d)$ of the solution $\bar{X}$ to (5.21) (in the sense of (5.20)) can be refined for certain intensity measures $\varrho(dtdx) \neq dtdx$. The measure states could have non-continuous densities, for instance. And this is indeed the case. If $\varrho(dtdx) = \varrho_1(dx)dt$ and the closed support $\mathrm{supp}(\varrho_1)$ of $\varrho_1(dx)$ has Lebesgue measure zero, then the states $\bar{X}_t(dx)$ of $\bar{X}$ possess Lebesgue densities $X_t(.)$, see [Del96]. Moreover, on the complement $\mathrm{supp}(\varrho_1)^c$ of $\mathrm{supp}(\varrho_1)$ these densities are smooth and solve the heat equation

$$\frac{\partial}{\partial t} X_t(x) = \frac{1}{2}\Delta X_t(x), \qquad (t,x) \in (0,\infty) \times \mathrm{supp}(\varrho_1)^c. \tag{5.22}$$

The validity of (5.22) is not surprising since the noise is only acting on $\mathrm{supp}(\varrho_1)$. Although the densities $X_t(.)$ are continuous on $\mathrm{supp}(\varrho_1)^c$, we do not expect them to be continuous on the whole $\mathbb{R}^d$. They should rather blow up (similar to (1.6)) when approaching the boundary of $\mathrm{supp}(\varrho_1)^c$; note that the noise is reinforced on $\mathrm{supp}(\varrho_1)$ in comparison to the case $\varrho(dtdx) = dtdx$. The singularity of $\varrho_1(dx)$ has a smoothing effect on the $\mathcal{M}_f(\mathbb{R}^d)$-valued solution $\bar{X}$ in the sense that the states $\bar{X}_t(dx)$ are absolutely continuous w.r.t. dx (in contrast to the case $\varrho(dtdx) = dtdx$ where the states are singular w.r.t. dx). At this point we stress the fact that the set $\mathrm{supp}(\varrho_1)$, i.e. the set where the noise acts, is rather thin in comparison to the "free of noise zone" $\mathrm{supp}(\varrho_1)^c$. This contrasts the case $\varrho(dtdx) = dtdx$ where the noise is acting everywhere.

In conclusion it should be mentioned that the situation becomes much better when considering SPDEs driven by so-called *colored* noises rather than by white ones. We are, however, not going into detail w.r.t. that issue. Roughly speaking, the difference between white and colored noises is that colored noises resign condition (*ii*) of Definition 5.1. In other words, the realizations of a colored noise on disjoint sets do not need to be independent. So, in some sense, colored noises are more "harmless" than white noises. In particular, the stochastic heat equation driven by a colored noise may have a jointly continuous solution also in higher dimensions $d \geq 2$, cf. [Dal99], [PZ00].

Standard references on SPDEs are [Wal86] and [DPZ92]. See also references therein.

6 Heat equation with inhomogeneous noise and singular drift ($d = 1$)

This chapter is devoted to solutions of SPDE (5.9) in the sense of Definition 5.20. We restrict to the space dimension $d = 1$; in Section 5.4 we mentioned that this restriction seems to be necessary. In Sections 6.1 and 6.2 we establish the equivalence of SPDE (5.9) in the sense of Definition 5.20 to both a certain martingale problem and a certain stochastic integral equation. These equivalences are essential for proving existence and uniqueness of solutions. In Section 6.3 we show the existence of strongly unique strong $C_{tem}(\mathbb{R})$-valued solutions for Lipschitz continuous coefficients, and in Section 6.4 we shall see that under some additional assumptions these solutions are non-negative. For (non-Lipschitz) continuous coefficients we obtain at least weak $C_{tem}(\mathbb{R})$-valued solutions, see Sections 6.5. Section 6.6 is devoted to the refinement of the state space $C_{tem}(\mathbb{R})$. More precisely, we show that, if the initial state η is from $C^+_{rap}(\mathbb{R})$, then the solution stays in $C^+_{rap}(\mathbb{R})$ forever. Recall that $a, b : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ are assumed to be continuous functions.

6.1 Corresponding martingale problem

In this section we show that any weak solution to SPDE (5.9) is a solution to the martingale problem posed in Definition 6.1 and vive versa (see Proposition 6.4). Recall that $\varrho(dtdx)$ and $\sigma(dtdx)$ were assumed to be Radon measures on $[0,\infty) \times \mathbb{R}$.

Definition 6.1 [MARTINGALE PROBLEM] *Let $\eta \in C_{tem}(\mathbb{R})$. A real-valued $(\mathcal{F}_t)$-predictable process $u = (u(t,x) : t \geq 0, x \in \mathbb{R}^d)$ on any filtered probability space $[\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}]$, where $(\mathcal{F}_t)$ satisfies the usual conditions, is said to be solution to the (a, b, η)-martingale problem if $(u(t,.) : t \geq 0)$ is $C_{tem}(\mathbb{R})$-valued continuous and if for every $\psi \in C_c^\infty(\mathbb{R})$:*

$$M_t(\psi) := \langle u(t,.), \psi\rangle - \langle \eta, \psi\rangle - \int_0^t \langle u(r,.), \frac{1}{2}\Delta\psi\rangle dr - \int_0^t \int_{\mathbb{R}} b(r,y,u(r,y))\psi(y)\sigma(drdy)$$

is a continuous square-integrable $(\mathcal{F}_t)$-martingale with quadratic variation process

$$\langle M(\psi)\rangle_t := \int_0^t \int_{\mathbb{R}} \psi^2(y) a^2(r,y,u(r,y))\varrho(drdy). \tag{6.1}$$

We say the solution is unique if any two solutions (which might be defined on different probability spaces) induce the same law on $C([0,\infty), C_{tem}(\mathbb{R}))$.

Before proving the (weak) equivalence of SPDE (5.9) and the (a, b, η)-martingale problem, we establish two lemmas.

Lemma 6.2 *If $u = [u, \Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}]$ is a $C_{tem}(\mathbb{R})$-valued solution to the (a, b, η)-martingale problem, then the family $(M_t(\psi)) \equiv (M_t(\psi) : \psi \in C_c^\infty(\mathbb{R}))$ of $\mathcal{M}_c^2$-martingales extends to a continuous orthogonal martingale measure, $M = (M_t(A) : t \geq 0, A \in \mathcal{A}(\mathbb{R}))$ with quadratic variation measure $\langle M\rangle(dtdx) = a^2(t,x,u(t,x))\varrho(dtdx)$.*

Proof For the moment fix $t > 0$. Set $\mu_t(dy) := \mathbb{E}[\int_0^t a^2(r,y,u(r,y))\varrho(drdy)]$ as well as

$$\|\psi\|_{2,\mu_t} := \mathbb{E}\Big[\int_0^t \int_{\mathbb{R}} \psi^2(y)a^2(r,y,u(r,y))\varrho(drdy)\Big]^{1/2}.$$

Note that $\|\psi\|_{2,\mu_t} < \infty$ for $\psi \in C_c^\infty(\mathbb{R})$ since the martingale $M_\cdot(\psi)$ is square-integrable. Let $\mathcal{M}_c^2([0,t])$ denote the space of square-integrable continuous martingales on the interval $[0,t]$ and recall the definition of $\|.\|_t$ from (3.13). The mapping

$$\psi \mapsto M_\cdot(\psi)|_{[0,t]}, \quad (C_c^\infty(\mathbb{R}), \|.\|_{2,\mu_t}) \to (\mathcal{M}_c^2([0,t]), \|.\|_t)$$

is linear and continuous. The linearity is evident and the continuity follows from

$$\|M_\cdot(\psi)\|_t = \mathbb{E}[M_t^2(\psi)]^{1/2} = \mathbb{E}[\langle M_\cdot(\psi)\rangle_t]^{1/2} = \|\psi\|_{2,\mu_t} \qquad \forall \psi \in C_c^\infty(\mathbb{R}).$$

Also, $C_c^\infty(\mathbb{R})$ is dense in $L^2(\mathbb{R},\mu_t)$ w.r.t. $\|.\|_{2,\mu_t}$[21] and $(\mathcal{M}_c^2([0,t]), \|.\|_t)$ is a Banach space. Thus $\psi \mapsto M_\cdot(\psi)|_{[0,t]}$ has a linear and continuous extension from $C_c^\infty(\mathbb{R})$ to $L^2(\mathbb{R},\mu_t)$. In particular,

$$M_\cdot(\psi)|_{[0,t]} = \lim_{n\to\infty} M_\cdot(\psi_n)|_{[0,t]} \qquad (\text{w.r.t. } \|.\|_t) \qquad \forall \psi \in L^2(\mathbb{R},\mu_t)$$

holds for every sequence $(\psi_n) \subset C_c^\infty(\mathbb{R})$ with $\lim_{n\to\infty} \|\psi_n - \psi\|_{2,\mu_t} = 0$.

Let $H_b(\mathbb{R})$ denote the space of functions $\psi \in B_b(\mathbb{R})$ such that $\psi(x) = 0$ $\forall x \notin B[0,r]$ for some $r > 0$. Clearly, $H_b(\mathbb{R}) \subset L^2(\mathbb{R},\mu_t)$ for all $t > 0$. From the first part of the proof we easily obtain, for every $\psi \in H_b(\mathbb{R})$, a martingale $M_\cdot(\psi) \in \mathcal{M}_c^2$ such that: For every $t > 0$,

$$M_\cdot(\psi)|_{[0,t]} = \lim_{n\to\infty} M_\cdot(\psi_n)|_{[0,t]} \qquad (\text{w.r.t. } \|.\|_t)$$

holds for every sequence $(\psi_n) \subset C_c^\infty(\mathbb{R})$ with $\lim_{n\to\infty} \|\psi_n - \psi\|_{2,\mu_t} = 0$. Also, $H_b(\mathbb{R}) \ni \psi \mapsto M_\cdot(\psi) \in \mathcal{M}_c^2$ is easily seen to be linear. Now, set $M_t(A) := M_t(\mathbf{1}_A)$ for every $A \in \mathcal{A}(\mathbb{R})$ and $t \geq 0$. To show that $M = (M_t(A) : t \geq 0, A \in \mathcal{A}(\mathbb{R}))$ is a continuous orthogonal martingale measure we have to verify $(i) - (iii)$ of Definition 5.8 and continuity of $M_\cdot(A)$ for every $A \in \mathcal{A}(\mathbb{R})$. Assertion (ii) and the continuity are already known. Assertion (i) can be shown easily, and so we omit the details. Note, however, that the $\mathbb{P}$-almost sure additivity of $A \mapsto M_t(A)$ follows from the linearity of $H_b(\mathbb{R}) \ni \psi \mapsto M_\cdot(\psi) \in \mathcal{M}_c^2$. To show (iii) we first prove that (6.1) also holds for every $\psi \in H_b(\mathbb{R})$. For it, it suffices to show that for every $s, t \geq 0$ and every bounded and $\mathcal{F}_t$-measurable h:

$$\mathbb{E}\Big[\Big(M_{t+s}^2(A) - M_t^2(A) - \int_t^{t+s}\int_A a^2(r,y,u(r,y))\varrho(drdy)\Big)h\Big] \;=\; 0 \qquad \mathbb{P}\text{-a.s.} \tag{6.2}$$

Choose $(\psi_n) \subset C_c^\infty(\mathbb{R})$ such that $\lim_{n\to\infty} \|\psi - \psi_n\|_{2,\mu_{t+s}} = 0$. Then we have in particular $\lim_{n\to\infty} \|\psi - \psi_n\|_{2,\mu_t} = 0$. By assumption we have for every $n \geq 1$ and every bounded and $\mathcal{F}_t$-measurable h:

$$\mathbb{E}\Big[\Big(M_{t+s}^2(\psi_n) - M_t^2(\psi_n) - \int_t^{t+s}\int_{\mathbb{R}} a^2(r,y,u(r,y))\psi_n^2(y)\varrho(drdy)\Big)h\Big] \;=\; 0.$$

[21] This can be shown with help of of Weierstrass's approximation theorem and Lemma 1.33 of [Kal97].

Consequently,

$$
\begin{aligned}
&\mathbb{E}\Big[\Big(M_{t+s}^2(\psi) - M_t^2(\psi) - \int_s^t \int_{\mathbb{R}} a^2(r,y,u(r,y))\psi^2(y)\varrho(drdy)\Big)h\Big] \\
&= \ \mathbb{E}\Big[\Big(\big(M_{t+s}^2(\psi) - M_{t+s}^2(\psi_n)\big) - \big(M_t^2(\psi) - M_t^2(\psi_n)\big) \qquad\qquad (6.3)\\
&\qquad - \int_s^t \int_{\mathbb{R}} a^2(r,y,u(r,y))\left(\psi^2(y) - \psi_n^2(y)\right)\varrho(drdy)\Big)h\Big].
\end{aligned}
$$

The first term on the r.h.s. of (6.3) converges to zero as $n \to \infty$ since

$$
\begin{aligned}
&\mathbb{E}\Big[\,\big|\big(M_{t+s}^2(\psi) - M_{t+s}^2(\psi_n)\big)\,h\big|\,\Big] \\
&\leq \ \|h\|_\infty \mathbb{E}\Big[\,|M_{t+s}(\psi) - M_{t+s}(\psi_n)|^2\Big] + 2\|h\|_\infty \mathbb{E}\Big[\,|(M_{t+s}(\psi) - M_{t+s}(\psi_n))M_{t+s}(\psi_n)|\,\Big] \\
&\leq \ \|h\|_\infty \mathbb{E}\Big[M_{t+s}^2(\psi - \psi_n)\Big] + 2\|h\|_\infty \mathbb{E}\Big[M_{t+s}^2(\psi - \psi_n)\Big]^{1/2}\mathbb{E}\Big[M_{t+s}^2(\psi_n)\Big]^{1/2} \\
&= \ \|h\|_\infty \mathbb{E}[\langle M(\psi - \psi_n)\rangle_{t+s}] + 2\|h\|_\infty \mathbb{E}[\langle M(\psi - \psi_n)\rangle_{t+s}]^{1/2}\mathbb{E}[\langle M(\psi_n)\rangle_{t+s}]^{1/2} \\
&= \ \|h\|_\infty \|\psi - \psi_n\|_{2,\mu_{t+s}}^2 + 2\|h\|_\infty \|\psi - \psi_n\|_{2,\mu_{t+s}}\|\psi_n\|_{2,\mu_{t+s}}.
\end{aligned}
$$

The second term on the r.h.s. of (6.3) tends to zero as $n \to \infty$ by the same arguments. The third term on the r.h.s. of (6.3) vanishes as $n \to \infty$ since

$$
\begin{aligned}
&\mathbb{E}\Big[\Big(\int_t^{t+s}\int_{\mathbb{R}} \left|\psi^2(y) - \psi^2(y)\right| a^2(r,y,u(r,y))\varrho(drdy)\Big)h\Big] \\
&\leq \ \|h\|_\infty \langle \mu_{t+s}, |\psi^2 - \psi_n^2|\rangle \\
&\leq \ \|h\|_\infty \Big(\|\psi - \psi_n\|_{2,\mu_{t+s}}^2 + 2\|\psi - \psi_n\|_{2,\mu_{t+s}}\|\psi_n\|_{2,\mu_{t+s}}\Big).
\end{aligned}
$$

On the whole, the l.h.s. of (6.3) vanishes, i.e. we obtain (6.2). Hence, (6.1) holds indeed for every $\psi \in H_b(\mathbb{R})$.

Now we are in the position to show *(iii)* of Definition 5.8. First of all note that

$$
B_t : (\phi, \psi) \mapsto \int_0^t \int_{\mathbb{R}} a^2(r,y,u(r,y))\phi(y)\psi(y)\varrho(drdy) \tag{6.4}
$$

is $\mathbb{P}$-almost surely a symmetric bilinear form on the linear space $H_b(\mathbb{R})$. We further note that for any $\phi, \psi \in H_b(\mathbb{R})$ and $\lambda \in \mathbb{R}$ there exist mean zero martingales N_λ and N such that:

$$
\langle M(\lambda\phi), M(\psi)\rangle_t = M(\lambda\phi)M(\psi) - N_\lambda = \lambda M(\phi)M(\psi) - N_\lambda
$$

and

$$
\lambda\langle M(\phi), M(\psi)\rangle_t = \lambda\Big(M(\phi)M(\psi) - N\Big) = \lambda M(\phi)M(\psi) - \lambda N.
$$

Then $\langle M(\lambda\phi), M(\psi)\rangle_t = \lambda\langle M(\phi), M(\psi)\rangle_t$ follows from the uniqueness of the Doob-Meyer decomposition (3.15). That is, $(\phi, \psi) \mapsto \langle M(\phi), M(\psi)\rangle_t$ is $\mathbb{P}$-almost surely a symmetric bilinear form, too (recall Lemma 3.27). Since the latter bilinear form coincides with the bilinear form B_t defined in (6.4) for $\phi = \psi$, and since any symmetric bilinear form can be recovered from its quadratic form through $B_t(\phi,\psi)=\frac{1}{4}[B(\phi+\psi,\phi+\psi)-B(\phi-\psi,\phi-\psi)]$, we obtain

$$\langle M(\phi),M(\psi)\rangle_t=\int_0^t\int_{\mathbb{R}} a^2(r,y,u(r,y))\phi(y)\psi(y)\varrho(drdy)\quad \mathbb{P}\text{-almost surely}$$

for every $\phi,\psi\in H_b(\mathbb{R})$ and $t\geq 0$. Hence, $M_.(\phi)$ and $M_.(\psi)$ are orthogonal whenever $\phi\psi\equiv 0$. In particular, $M_.(A)$ and $M_.(A')$ are orthogonal for any disjoint $A,A'\in\mathcal{A}(\mathbb{R})$. That is, *(iii)* of Definition 5.8 holds.

Finally, the statement on the quadratic variation measure is an easy consequence of the fact that (6.1) holds for every $\psi\in H_b(\mathbb{R})$. □

Recall that $H_b(\mathbb{R})$ is the space of functions $\psi\in B_b(\mathbb{R})$ such that $\psi(x)=0\ \forall x\notin B[0,r]$ for some $r>0$. In the previous proof we extended the map $\psi\mapsto M_.(\psi)\in\mathcal{M}_c^2$ from $C_c^\infty(\mathbb{R})$ to $H_b(\mathbb{R})$, where $M_.(\psi)$ is defined as in Definition 6.1 for $\psi\in C_c^\infty(\mathbb{R})$.

Lemma 6.3 *Let M be the continuous orthogonal martingale measure from Lemma 6.2. Then we have for every $\psi\in H_b(\mathbb{R})$:*

$$M_t(\psi)=\int_0^t\int_{\mathbb{R}}\psi(y)M(drdy)\qquad \forall t\geq 0,\ \mathbb{P}\text{-almost surely.}\tag{6.5}$$

Proof First of all note that $H_b(\mathbb{R})$ is a subspace of $\mathcal{P}^2(M)$. According to Proposition 5.12 we may choose a sequence (ψ_n) of (deterministic and time-independent) simple functions such that $|[\psi-\psi_n]|_t\to 0$ as $n\to\infty$. We clearly have $\|M_.(\psi)-\int_0^. \int\psi(y)M(drdy)\|\leq \|M_.(\psi)-M_.(\psi_n)\|+\|M_.(\psi_n)-\int_0^.\int\psi(y)M(drdy)\|$. The first summand tends to zero since $\|M_.(\psi)-M_.(\psi_n)\|_t\leq|[\psi-\psi_n]|_t$ for all $t\geq 0$. The second summand tends to zero by the construction of $f\cdot M$ and

$$M_t(\psi_n)=\sum_{i=1}^{m_n}c_i^n M_t(\mathbf{1}_{B_i^n})=\sum_{i=1}^{m_n}c_i^n M_t(B_i^n)=\int_0^t\int_{\mathbb{R}}\psi_n(y)M(drdy)\quad\Big(\text{if }\psi_n=\sum_{i=1}^{m_n}c_i^n\mathbf{1}_{B_i^n}\Big)$$

w.r.t. $\|.\|$; recall the linearity of $H_b(\mathbb{R})\ni\psi\mapsto M_.(\psi)\in\mathcal{M}_c^2$. Hence, we obtain (6.5). □

Proposition 6.4 [EQUIVALENCE OF SPDE AND MP] *Assume $\varrho(dtdx)\in\mathcal{M}([0,\infty)\times\mathbb{R})$ is as in Example 5.10. Then every weak $C_{tem}(\mathbb{R})$-valued solution to SPDE (5.9) with initial condition $\eta\in C_{tem}(\mathbb{R})$ in the sense of Definition 5.20 is a $C_{tem}(\mathbb{R})$-valued solution to the (a,b,η)-martingale problem, and vice versa.*

Proof Let $u=[u,\Omega,\mathcal{F},(\mathcal{F}_t),\mathbb{P},W^\varrho]$ be a weak $C_{tem}(\mathbb{R})$-valued solution to SPDE (5.9) with initial condition $\eta\in C_{tem}(\mathbb{R})$. Then, for all $\psi\in C_c^\infty(\mathbb{R})$,

$$\begin{aligned}M_t(\psi)&:=\int_0^t\int_{\mathbb{R}}a(r,y,u(r,y))\psi(y)W^\varrho(drdy)\\&=\ \langle u(t,.),\psi\rangle-\langle\eta,\psi\rangle-\int_0^t\langle u(r,.),\frac{1}{2}\Delta\psi\rangle dr-\int_0^t\int_{\mathbb{R}}b(r,y,u(r,y))\psi(y)\sigma(drdy)\end{aligned}$$

provides a continuous square-integrable $(\mathcal{F}_t)$-martingale with quadratic variation process

$$\langle M(\psi)\rangle_t = \int_0^t \int_{\mathbb{R}} a^2(r,y,u(r,y))\psi^2(y)\langle W^\varrho\rangle(drdy) = \int_0^t \int_{\mathbb{R}} a^2(r,y,u(r,y))\psi^2(y)\varrho(drdy).$$

Conversely, let $u = [u, \Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}]$ be a $C_{tem}(\mathbb{R})$-valued solution of the (a,b,η)-martingale problem. According to Lemma 6.2, the family $(M_t(\psi))$ of $\mathcal{M}_c^2$-martingales extends to a continuous orthogonal martingale measure, M, with quadratic variation measure $\langle M\rangle(dtdx) = a^2(t,x,u(t,x))\varrho(dtdx)$. We also may and do pick a continuous orthogonal martingale measure, $\tilde{W}^\varrho$, with quadratic variation measure $\langle \tilde{W}^\varrho\rangle(dtdx) = \varrho(dtdx)$ (recall Example 5.10). W.l.o.g. we assume that $\tilde{W}^\varrho$ is independent of M. If necessary, consider an enlargement of u's domain $[\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}]$. For $t \geq 0$ and $A \in \mathcal{A}(\mathbb{R})$ set

$$\begin{aligned} W_t^\varrho(A) := \int_0^t \int_{\mathbb{R}} \mathbf{1}_A(y)\mathbf{1}_{a(r,y,u(r,y))\neq 0}\frac{1}{a(r,y,u(r,y))}M(drdy) \\ + \int_0^t \int_{\mathbb{R}} \mathbf{1}_A(y)\mathbf{1}_{a(r,y,u(r,y))=0}\tilde{W}^\varrho(drdy). \end{aligned} \tag{6.6}$$

The admissibility of the integrand of the stochastic integral w.r.t. $\tilde{W}^\varrho$ is easy to see. The admissibility of the integrand of the stochastic integral w.r.t. M is not so obvious. However, $\mathbf{1}_A(y)\mathbf{1}_{a(r,y,u(r,y))\neq 0}a^{-1}(r,y,u(r,y))$ is $(\mathcal{F}_t)$-predictable since u is. Moreover,

$$\begin{aligned} &\mathbb{E}\Big[\int_0^t \int_{\mathbb{R}} \Big(\mathbf{1}_A(y)\mathbf{1}_{a(r,y,u(r,y))\neq 0}\frac{1}{a(r,y,u(r,y))}\Big)^2 \langle M\rangle(drdy)\Big] \\ &= \mathbb{E}\Big[\int_0^t \int_A \mathbf{1}_{a(r,y,u(r,y))\neq 0}\frac{1}{a^2(r,y,u(r,y))}a^2(r,y,u(r,y))\varrho(drdy)\Big] \\ &\leq \int_0^t \int_A \varrho(drdy) < \infty \qquad \forall t > 0. \end{aligned}$$

Hence, we have the desired admissibility, i.e. $\mathbf{1}_A(y)\mathbf{1}_{a(r,y,u(r,y))\neq 0}a^{-1}(r,y,u(r,y)) \in \mathcal{P}^2(M)$. Note that the two martingales on the r.h.s. of (6.6) are orthogonal; this follows from Lemma 5.17 and the independence of $\tilde{W}^\varrho$ of M. Then it is easy to verify that $W^\varrho = (W_t^\varrho(A) : t \geq 0, A \in \mathcal{A}(\mathbb{R}))$ provides a continuous orthogonal martingale measure with quadratic variation measure $\langle W^\varrho\rangle(dtdx) = \varrho(dtdx)$. The statement on the quadratic variation measure follows from

$$\begin{aligned} \langle W^\varrho\rangle((s,t]\times A) &= \Big\langle \int_s^\cdot \int_{\mathbb{R}} \mathbf{1}_A(y)W^\varrho(drdy)\Big\rangle_t \\ &= \Big\langle \int_s^\cdot \int_{\mathbb{R}} \mathbf{1}_A(y)\mathbf{1}_{a(r,y,u(r,y))\neq 0}\frac{1}{a(r,y,u(r,y))}M(drdy)\Big\rangle_t \\ &\quad + \Big\langle \int_s^\cdot \int_{\mathbb{R}} \mathbf{1}_A(y)\mathbf{1}_{a(r,y,u(r,y))=0}\tilde{W}^\varrho(drdy)\Big\rangle_t \\ &= \int_s^t \int_A \mathbf{1}_{a(r,y,u(r,y))\neq 0}\frac{1}{a(r,y,u(r,y))}\langle M\rangle(drdy) + \int_s^t \int_A \mathbf{1}_{a(r,y,u(r,y))=0}\langle\tilde{W}^\varrho\rangle(drdy) \\ &= \int_s^t \int_A \mathbf{1}_{a(r,y,u(r,y))\neq 0}\frac{1}{a(r,y,u(r,y))}a(r,y,u(r,y))\varrho(drdy) \end{aligned}$$

$$
+\int_s^t \int_A \mathbf{1}_{a(r,y,u(r,y))=0}\varrho(drdy)
$$

$$
= \int_s^t \int_A \varrho(drdy) \;=\; \varrho((s,t]\times A) \quad \mathbb{P}\text{-a.s., for all } 0\le s\le t \text{ and } A\in\mathcal{A}(\mathbb{R}).
$$

By Lemma 6.3 we also obtain

$$
\begin{aligned}
M_t(\psi) &= \int_0^t \int_{\mathbb{R}} \psi(y)M(drdy) \\
&= \int_0^t \int_{\mathbb{R}} \psi(y)a(r,y,u(r,y))\frac{1}{a(r,y,u(r,y))}M(drdy) \\
&= \int_0^t \int_{\mathbb{R}} \psi(y)a(r,y,u(r,y))W^{\varrho}(drdy) \qquad \mathbb{P}\text{-almost surely}
\end{aligned}
$$

for every $t>0$ and $\psi\in C_c^\infty(\mathbb{R})$. Combining this with the assumption eventually yields

$$
\begin{aligned}
\langle u(t,.),\psi\rangle &= \langle \eta,\psi\rangle + \int_0^t \langle u(r,.),\frac{1}{2}\Delta\psi\rangle dr \\
&+\int_0^t \int_{\mathbb{R}^d} b(r,y,u(r,y))\psi(y)\sigma(drdy) + \int_0^t \int_{\mathbb{R}^d} a(r,y,u(r,y))\psi(y)W^{\varrho}(drdy)
\end{aligned}
$$

for all $t\ge 0$ and $\psi\in C_c^\infty(\mathbb{R}^d)$, $\mathbb{P}$-almost surely. This completes the proof. □

6.2 Corresponding stochastic integral equation

In this section we show that any strong solution to SPDE (5.9) is a solution to SIE (6.7) and vive versa (see Proposition 6.7). For it we assume that $\varrho(dtdx)$ satisfies condition (A) and $\sigma(dtdx)$ satisfies conditions (B). These conditions were introduced in Definitions 2.21 and 2.22. Recall that $W^{\varrho}=[W^{\varrho},\Omega,\mathcal{F},(\mathcal{F}_t),\mathbb{P}]$ denotes a continuous orthogonal martingale measure with quadratic variation measure $\langle M\rangle(dtdx)=\varrho(dtdx)$ and that (P_t) denotes the heat semigroup defined at the end of Section 4.2.

Definition 6.5 [STOCHASTIC INTEGRAL EQUATION] *Given the continuous orthogonal martingale measure $W^{\varrho}=[W^{\varrho},\Omega,\mathcal{F},(\mathcal{F}_t),\mathbb{P}]$, a real-valued $(\mathcal{F}_t)$-predictable process $u=(u(t,x):t\ge 0,x\in\mathbb{R})$ on $[\Omega,\mathcal{F},\mathbb{P}]$ is said to be $C_{tem}(\mathbb{R})$-valued solution to SIE (6.7) with initial condition $\eta\in C_{tem}(\mathbb{R})$ if $(u(t,.):t\ge 0)$ is $C_{tem}(\mathbb{R})$-valued continuous and*

$$
\begin{aligned}
u(t,x) &= P_t\eta(x) + \int_0^t \int_{\mathbb{R}} p_{t-r}(x,y)b(r,y,u(r,y))\sigma(drdy) \\
&\quad + \int_0^t \int_{\mathbb{R}} p_{t-r}(x,y)a(r,y,u(r,y))W^{\varrho}(drdy)
\end{aligned} \tag{6.7}
$$

holds for all $t\ge 0$ and $x\in\mathbb{R}$, $\mathbb{P}$-almost surely. We say the solution is unique if any two solutions w.r.t. a given martingale measure $W^{\varrho}=[W^{\varrho},\Omega,\mathcal{F},(\mathcal{F}_t),\mathbb{P}]$ are indistinguishable.

Let $C^2_{rap}(\mathbb{R})$ be the space of functions $\psi \in C_{rap}(\mathbb{R})$ with $\psi' := \frac{d}{dx}\psi, \psi'' := \frac{d^2}{dx^2}\psi \in C_{rap}(\mathbb{R})$, equipped with the metric $d_{rap,2}(\phi,\psi) := d_{rap}(\phi,\psi) + d_{rap}(\phi',\psi') + d_{rap}(\phi'',\psi'')$.

Lemma 6.6 *Assume for every $T > 0$ there exists a finite constant $c_T > 0$ such that*

$$|a(t,x,u)| + |b(t,x,u)| \leq c_T(1+|u|) \tag{6.8}$$

holds for all $t \leq T$ and $x, u \in \mathbb{R}$. Let $u(.,.)$ be a strong $C_{tem}(\mathbb{R})$-valued solution to SPDE (5.9) with initial condition $\eta \in C_{tem}(\mathbb{R})$ in the sense of Definition 5.20. Then (5.15) even holds for all $\psi \in C^2_{rap}(\mathbb{R})$.

Proof Note that $C_c^\infty(\mathbb{R})$ is dense in $C^2_{rap}(\mathbb{R})$ w.r.t. $d_{rap,2}$. Hence, for every $\psi \in C^2_{rap}(\mathbb{R})$, we can choose $(\psi_n) \subset C_c^\infty(\mathbb{R})$ such that $\lim_{n\to\infty} d_{rap,2}(\psi,\psi_n) = 0$. In order to prove the statement of the lemma it is enough to show that the terms in (5.15) with ψ replaced by ψ_n converge $\mathbb{P}$-almost surely to the corresponding terms in (5.15) with $\psi \in C^2_{rap}(\mathbb{R})$. The convergence of the l.h.s. and the first three summands on the r.h.s. follows easily by $d_{rap}(\psi,\psi_n) \to 0$ and $d_{rap}(\phi'',\psi_n'') \to 0$. The least obvious convergence is the one of the stochastic integrals. We only show that convergence in detail. For every $K > 0$ and arbitrary $\lambda > 0$ we define the stopping time $\tau_K := \inf\{t > 0 : |u_t|_{(-\lambda)} \geq K\}$. Using (6.8) we obtain

$$\begin{aligned}
&\mathbb{E}\Big[\Big|\int_0^{t\wedge\tau_K}\int_{\mathbb{R}} a(r,y,u(r,y))\psi(y)W^{\varrho}(drdy) - \int_0^{t\wedge\tau_K}\int_{\mathbb{R}} a(r,y,u(r,y))\psi_n(y)W^{\varrho}(drdy)\Big|^2\Big]\\
&= \mathbb{E}\Big[\int_0^{t\wedge\tau_K}\int_{\mathbb{R}} a^2(r,y,u(r,y))\Big(\psi(y)-\psi_n(y)\Big)^2\varrho(drdy)\Big]\\
&\leq \mathbb{E}\Big[\int_0^{t\wedge\tau_K}\int_{\mathbb{R}} c_T^2(1+u(r,y))^2\Big(\psi(y)-\psi_n(y)\Big)^2\varrho(drdy)\Big]\\
&= \mathbb{E}\Big[\int_0^{t\wedge\tau_K}\int_{\mathbb{R}} c_T^2(1+u(r,y))^2 e^{-2\lambda|y|}e^{-\lambda|y|}\Big((\psi(y)-\psi_n(y))e^{3\lambda|y|/2}\Big)^2\varrho(drdy)\Big]\\
&\leq c_T^2(1+K)^2|\psi-\psi_n|^2_{(3\lambda/2)}\int_0^t\int_{\mathbb{R}} e^{-\lambda|y|}\varrho(drdy) \;\to\; 0 \quad \text{as } n\to\infty
\end{aligned}$$

for every $K > 0$. Consequently, for every $K > 0$, (5.15) holds for all $\psi \in C^2_{rap}(\mathbb{R})$ and $t \leq \tau_K$, $\mathbb{P}$-almost surely. However, u is continuous, hence τ_K converges to ∞ as $K \uparrow \infty$. This implies (5.15) for all $\psi \in C^2_{rap}(\mathbb{R})$ and $t \geq 0$, $\mathbb{P}$-almost surely. □

Proposition 6.7 [EQUIVALENCE OF SPDE AND SIE] *Assume $\varrho(dtdx)$ satisfies condition* (A) *and $\sigma(dtdx)$ satisfies condition* (B). *If the coefficients a and b satisfy (6.8), then every strong $C_{tem}(\mathbb{R})$-valued solution to SPDE (5.9) with initial condition $\eta \in C_{tem}(\mathbb{R})$ in the sense of Definition 5.20 is a $C_{tem}(\mathbb{R})$-valued solution to SIE (6.7) with initial condition η, and vice versa.*

Proof We adapt arguments of Shiga ([Shi94]). Let u be a solution to SIE (6.7). Since $\frac{1}{2}\Delta$ is the generator of (P_t), we obtain by (3.21) that $\|\int_0^t P_s\Delta\psi(.)ds - (P_t\psi(.)-\psi(.))\|_\infty = 0$

holds for every $\psi \in C_0^2(\mathbb{R})$. Thus, using the classical and the stochastic Fubini theorem (Proposition 5.19), we obtain for every $\psi \in C_c^\infty(\mathbb{R}) \subset C_0^2(\mathbb{R})$:

$$
\begin{aligned}
&\int_0^t \langle u(s,.), \frac{1}{2}\Delta\psi\rangle ds \qquad\qquad (6.9)\\
=\ & \int_0^t \langle \eta, P_s \frac{1}{2}\Delta\psi\rangle ds + \int_0^t \int_{\mathbb{R}} \int_0^s \int_{\mathbb{R}} p_{s-r}(x,y) b(r,y,u(r,y))\sigma(drdy)\frac{1}{2}\Delta\psi(x) dxds \\
&\quad + \int_0^t \int_{\mathbb{R}} \int_0^s \int_{\mathbb{R}} p_{s-r}(x,y) a(r,y,u(r,y)) W^{\varrho}(drdy)\frac{1}{2}\Delta\psi(x) dxds \\
=\ & \langle \eta, \int_0^t P_s \frac{1}{2}\Delta\psi(.) ds\rangle + \int_0^t \int_0^s \int_{\mathbb{R}} (P_{s-r}\frac{1}{2}\Delta\psi)(y) b(r,y,u(r,y))\sigma(drdy) ds \\
&\quad + \int_0^t \int_0^s \int_{\mathbb{R}} (P_{s-r}\frac{1}{2}\Delta\psi)(y) a(r,y,u(r,y)) W^{\varrho}(drdy) ds \\
=\ & \langle \eta, P_t\psi - \psi\rangle + \int_0^t \int_{\mathbb{R}} \int_r^t (P_{s-r}\frac{1}{2}\Delta\psi)(y)\ ds\ b(r,y,u(r,y))\sigma(drdy) \\
&\quad + \int_0^t \int_{\mathbb{R}} \int_r^t (P_{s-r}\frac{1}{2}\Delta\psi)(y)\ ds\ a(r,y,u(r,y)) W^{\varrho}(drdy) \\
=\ & \langle \eta, P_t\psi - \psi\rangle + \int_0^t \int_{\mathbb{R}} [P_{t-r}\psi(y) - \psi(y)] b(r,y,u(r,y))\sigma(drdy) \\
&\quad + \int_0^t \int_{\mathbb{R}} [P_{t-r}\psi(y) - \psi(y)] a(r,y,u(r,y)) W^{\varrho}(drdy).
\end{aligned}
$$

and

$$
\begin{aligned}
&\langle u(t,.), \psi\rangle \qquad\qquad (6.10)\\
=\ & \langle \eta, P_t\psi\rangle + \int_{\mathbb{R}} \int_0^t \int_{\mathbb{R}} p_{t-r}(x,y) b(r,y,u(r,y))\sigma(drdy)\psi(x) dx \\
&\quad + \int_{\mathbb{R}} \int_0^t \int_{\mathbb{R}} p_{t-r}(x,y) a(r,y,u(r,y)) W^{\varrho}(drdy)\psi(x) dx \\
=\ & \langle \eta, P_t\psi\rangle + \int_0^t \int_{\mathbb{R}} P_{t-r}\psi(y) b(r,y,u(r,y))\sigma(drdy) \\
&\quad + \int_0^t \int_{\mathbb{R}} P_{t-r}\psi(y) a(r,y,u(r,y)) W^{\varrho}(drdy).
\end{aligned}
$$

Subtracting (6.9) from (6.10) we see that u is a strong solution to SPDE (5.9).

For the proof of the converse direction we introduce the space

$$
\begin{aligned}
C^{1,2}_{rap}([0,\infty)\times\mathbb{R}) \ :=\ \Big\{ & f \in C^{1,2}([0,\infty)\times\mathbb{R}) \ : \\
& (f(t,.) : t \geq 0) \text{ is } C^2_{rap}(\mathbb{R})\text{-valued continuous}, \\
& (\frac{\partial}{\partial t} f(t,.) : t \geq 0) \text{ is } C_{rap}(\mathbb{R})\text{-valued continuous} \Big\}.
\end{aligned}
$$

Recall that $C^2_{rap}(\mathbb{R})$ is the space of functions $\psi \in C_{rap}(\mathbb{R})$ satisfying $\frac{d}{dx}\psi$, $\frac{d^2}{dx^2}\psi \in C_{rap}(\mathbb{R})$, equipped with the metric $d_{rap,2}(\phi,\psi) = d_{rap}(\phi,\psi) + d_{rap}(\phi',\psi') + d_{rap}(\phi'',\psi'')$. Let u be a strong $C_{tem}(\mathbb{R})$-valued solution to SPDE (5.9), $t \geq 0$ and $\pi = \{0 = t_0 \leq t_1 \cdots \leq t_N = t\}$ be a partition of $[0,t]$. Set $|\pi| := \max_{0 \leq i \leq N} |t_i - t_{i-1}|$ as well as $\bar{\pi}(r) := t_i$ and $\underline{\pi}(r) := t_{i-1}$ for $r \in [t_{i-1}, t_i)$. Also, $u_t := u(t,.)$ for $t \geq 0$. By means of Lemma 6.6 we obtain for every $f \in C^{1,2}_{rap}([0,\infty) \times \mathbb{R})$:

$$
\begin{aligned}
&\langle u_t, f_t \rangle \qquad (6.11)\\
&= \langle \eta, f_0 \rangle + \sum_{i=1}^{N} \Big\{ \langle u_{t_i}, f_{t_i} - f_{t_{i-1}} \rangle + \langle u_{t_i} - u_{t_{i-1}}, f_{t_{i-1}} \rangle \Big\} \\
&= \langle \eta, f_0 \rangle + \sum_{i=1}^{N} \Big\{ \langle u_{t_i}, \int_{t_{i-1}}^{t_i} \frac{\partial}{\partial r} f_r dr \rangle + \int_{t_{i-1}}^{t_i} \langle u_r, \frac{1}{2} \Delta f_{t_{i-1}} \rangle dr + \\
&\quad \int_{t_{i-1}}^{t_i} \int_{\mathbb{R}} b(r,y,u(r,y)) f_{t_{i-1}}(y) \sigma(drdy) + \int_{t_{i-1}}^{t_i} \int_{\mathbb{R}} a(r,y,u(r,y)) f_{t_{i-1}}(y) W^{\varrho}(drdy) \Big\} \\
&= \langle \eta, f_0 \rangle + \Big(\int_0^t \langle u_{\bar{\pi}(r)}, \frac{\partial}{\partial r} f_r \rangle dr + \int_0^t \langle u_r, \frac{1}{2} \Delta f_{\underline{\pi}(r)} \rangle dr \Big) + \\
&\quad \int_0^t \int_{\mathbb{R}} b(r,y,u(r,y)) f_{\underline{\pi}(r)}(y) \sigma(drdy) + \int_0^t \int_{\mathbb{R}} a(r,y,u(r,y)) f_{\underline{\pi}(r)}(y) W^{\varrho}(drdy).
\end{aligned}
$$

Letting $|\pi| \to 0$ leads to

$$
\begin{aligned}
\langle u_t, f_t \rangle &= \langle \eta, f_0 \rangle + \int_0^t \langle u_r, \frac{\partial}{\partial r} f_r + \frac{1}{2} \Delta f_r \rangle dr \qquad (6.12)\\
&\quad + \int_0^t \int_{\mathbb{R}} b(r,y,u(r,y)) f_r(y) \sigma(drdy) + \int_0^t \int_{\mathbb{R}} a(r,y,u(r,y)) f_r(y) W^{\varrho}(drdy)
\end{aligned}
$$

for all $t \geq 0$ and $f \in C^{1,2}_{rap}([0,\infty) \times \mathbb{R})$, $\mathbb{P}$-almost surely. Indeed: The second and the third summand on the very r.h.s. of (6.11) converge $\mathbb{P}$-almost surely to the second and the third summand on the r.h.s. of (6.12), respectively, since $(u_r : r \geq 0)$ is $C_{tem}(\mathbb{R})$-valued continuous and $(f_r : r \geq 0)$, $(f'_r : r \geq 0)$ and $(f''_r : r \geq 0)$ are $C_{rap}(\mathbb{R})$-valued continuous (uniformly on compacts). To show the convergence of the stochastic integrals we define, as in the proof of Lemma 6.6, the stopping time $\tau_K := \inf\{t > 0 : |u_t|_{(-\lambda)} \geq K\}$ for every $K > 0$. By means of (6.8) we obtain for every $\lambda > 0$ and $K > 0$:

$$
\begin{aligned}
&\mathbb{E}\Big[\Big| \int_0^{t \wedge \tau_K} \int_{\mathbb{R}} a(r,y,u(r,y)) \Big(f_{\underline{\pi}(r)}(y) - f_r(y) \Big) W^{\varrho}(drdy) \Big|^2 \Big] \qquad (6.13)\\
&= \mathbb{E}\Big[\int_0^{t \wedge \tau_K} \int_{\mathbb{R}} a^2(r,y,u(r,y)) \Big(f_{\underline{\pi}(r)}(y) - f_r(y) \Big)^2 \varrho(drdy) \Big] \\
&\leq \mathbb{E}\Big[\int_0^{t \wedge \tau_K} \int_{\mathbb{R}} c_T^2 (1 + u(r,y))^2 e^{-3\lambda|y|} e^{3\lambda|y|} \Big(f_{\underline{\pi}(r)}(y) - f_r(y) \Big)^2 \varrho(drdy) \Big] \\
&\leq \mathbb{E}\Big[\int_0^{t \wedge \tau_K} \int_{\mathbb{R}} c_T^2 (1 + u(r,y))^2 e^{-3\lambda|y|} \Big(\sup_{r \leq t} \sup_{z \in \mathbb{R}} e^{3\lambda|z|/2} \Big| f_{\underline{\pi}(r)}(z) - f_r(z) \Big| \Big)^2 \varrho(drdy) \Big] \\
&\leq c_T^2 (1+K)^2 \sup_{r \leq t} \Big| f_{\underline{\pi}(r)} - f_r \Big|^2_{(3\lambda/2)} \int_0^t \int_{\mathbb{R}} e^{-\lambda|y|} \varrho(drdy).
\end{aligned}
$$

The latter estimate tends to zero as $|\pi| \downarrow 0$ since $(f_r : r \geq 0)$ is $C_{rap}(\mathbb{R})$-valued continuous (uniformly on compacts). Hence, for every $K > 0$, (6.12) holds for all $t \leq \tau_K$, $\mathbb{P}$-almost surely. Since u is continuous, τ_K converges to ∞ as $K \uparrow \infty$ and so (6.12) holds indeed for all $t \geq 0$, $\mathbb{P}$-almost surely. Now, set $f(r,y) = p_{t-r+\epsilon}(x,y)$ for all $r \in [0,t]$. Then f belongs to $C^{1,2}_{rap}([0,t] \times \mathbb{R})$ and (6.12) turns into

$$\begin{aligned} \langle u_t, p_\epsilon(x,.)\rangle \;&=\; \langle \eta, p_{t+\epsilon}(x,.)\rangle + \int_0^t \langle u_r, \frac{\partial}{\partial r} P_{t-r} p_\epsilon(x,.) + \frac{1}{2}\Delta P_{t-r} p_\epsilon(x,.)\rangle dr + \\ &\int_0^t \int_{\mathbb{R}} b(r,y,u(r,y)) p_{t-r+\epsilon}(x,y) \sigma(drdy) + \int_0^t \int_{\mathbb{R}} a(r,y,u(r,y)) p_{t-r+\epsilon}(x,y) W^\varrho(drdy). \end{aligned} \tag{6.14}$$

The second term on the r.h.s. of (6.14) vanishes since $\frac{1}{2}\Delta$ is the generator of (P_t). Letting $\epsilon \downarrow 0$ yields (6.7). (In order to show the convergence of the last two integrals on the r.h.s. of (6.14) to the two integrals on the r.h.s. of (6.7) one can proceed as in (6.13) where one has to use Lemma 4.7, respectively 4.6, instead of $\sup_{r\leq t} |f_{\underline{\pi}(r)} - f_r|_{(3\lambda/2)} \to 0$.) □

6.3 Strong solutions (Lipschitz continuous coefficients)

Here we are going to establish strongly unique strong solutions to SPDE (5.9) in the sense of Definition 5.20, provided the coefficients are Lipschitz continuous and grow at most linearly. Conditions (A) and (B) were introduced in Definitions 2.21 and 2.22.

Theorem 6.8 [UNIQUE STRONG $C_{tem}(\mathbb{R})$-VALUED SOLUTION] *Let a and b be continuous. Assume for every $T > 0$ there are finite constants $c_T, L_T > 0$ such that*

$$|a(t,x,u)| + |b(t,x,u)| \leq c_T(1 + |u|) \tag{6.15}$$

$$|a(t,x,u) - a(t,x,u')| + |b(t,x,u) - b(t,x,u')| \leq L_T |u - u'| \tag{6.16}$$

hold for all $t \leq T$ and $x, u, u' \in \mathbb{R}$. Let $\eta \in C_{tem}(\mathbb{R})$, $\varrho(dtdx)$ satisfy condition (A) *with α_1, α_2 and $\sigma(dtdx)$ satisfy condition* (B) *with β_1, β_2. Then SPDE (5.9) with initial condition η has a strongly unique strong $C_{tem}(\mathbb{R})$-valued solution. This solution is locally jointly Hölder-γ-continuous for all $\gamma \in (0, \frac{\alpha}{2} \wedge \beta)$ where $\alpha := \frac{\alpha_1}{2} + \alpha_2 - 1$, $\beta := \frac{\beta_1}{2} + \beta_2 - 1/2$.*

Proof We shall prove that SIE (6.7) has a unique solution and so, by Proposition 6.7, the same is true for SPDE (5.9). Given the continuous orthogonal martingale measure $W^\varrho = [W^\varrho, \Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}]$, let $\mathcal{P}$ be the space of $(\mathcal{F}_t)$-predictable functions $u : [0,\infty) \times \mathbb{R} \times \Omega \to \mathbb{R}$ with $\|u\|_{\lambda,T,m} < \infty$ for all $\lambda, T > 0$ and $m \geq 1$, where:

$$\|u\|_{\lambda,T,m} := \Big(\sup_{t\leq T} \sup_{x\in\mathbb{R}} e^{-\lambda|x|} \mathbb{E}\Big[|u(t,x)|^{2m}\Big]\Big)^{\frac{1}{2m}}.$$

We identify $u, u' \in \mathcal{P}$ if $u(t,x) = u'(t,x)$ holds $\mathbb{P}$-almost surely for every fixed $(t,x) \in [0,\infty) \times \mathbb{R}$. Then $d_{\mathcal{P}}(f,f') = \sum_{k,l,m=1}^\infty 2^{-(k+l+m)}(1 \wedge \|f - f'\|_{\frac{1}{k},l,m})$ provides a metric on

$\mathcal{P}$ w.r.t. which $\mathcal{P}$ is complete. For the sake of a Picard-Lindelöf iteration we introduce the functional

$$\begin{aligned}\Phi(u)(t,x) \;&:=\; P_t\eta(x) \;+\; \Phi_2(u)(t,x) \;+\; \Phi_3(u)(t,x) \\ &:=\; P_t\eta(x) + \int_0^t \int_{\mathbb{R}} p_{t-r}(x,y) b(r,y,u(r,y))\sigma(drdy) \\ &\qquad + \int_0^t \int_{\mathbb{R}} p_{t-r}(x,y) a(r,y,u(r,y)) W^{\varrho}(drdy).\end{aligned}$$

For $u \in \mathcal{P}$ the stochastic integral is well-defined since the integrand is admissible w.r.t. W^{ϱ}. This is guaranteed by the following estimate:

$$\begin{aligned}&\mathbb{E}\Big[\int_0^t \int_{\mathbb{R}} p_{t-r}^2(x,y) a^2(r,y,u(r,y))\langle W^{\varrho}\rangle(drdy)\Big] \\ &\leq\; \int_0^t \int_{\mathbb{R}} p_{t-r}^2(x,y)\, e^{|y|}e^{-|y|}\, c\,\mathbb{E}[(1+|u(r,y)|)^2]\varrho(drdy) \\ &\leq\; c'(1+\|u\|_{1,t,1}^2)\frac{1}{2\pi}\int_0^t \frac{1}{t-r}\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{(t-r)}}\, e^{|y|}\varrho_1(r,dy)\varrho_2(dr) \\ &\leq\; c_t'\int_0^t \frac{1}{t-r} e^{|x|}(t-r)^{\alpha_1/2}\varrho_2(dr) \leq c_t\, t^{\alpha_1/2+\alpha_2-1}\, e^{|x|} \;<\; \infty \quad \forall t \geq 0\end{aligned}$$

for which we used (6.15), Lemma 4.2(i) $\Rightarrow$ (iv) and Lemma 4.4(i). In Step 2 below we will also see that $\Phi(\mathcal{P}) \subset \mathcal{P}$.

Step 1. We first prove that $\Phi(u)$ may be assumed to be $C_{tem}(\mathbb{R})$-valued continuous and $(\mathcal{F}_t)$-predictable whenever $u \in \mathcal{P}$. Using Hölder's inequality ($\frac{2m-1}{2m} + \frac{1}{2m} = 1$), (6.15) and Lemma 4.6 we get for all $0 \leq t \leq t' \leq T$ and $x, x' \in \mathbb{R}$:

$$\begin{aligned}&\mathbb{E}\Big[\Big|\Phi_2(u)(t,x) - \Phi_2(u)(t',x')\Big|^{2m}\Big] \\ &\leq\; \mathbb{E}\Big[\Big|\int_0^{t'}\int_{\mathbb{R}} (p_{t-r}(x,y) - p_{t'-r}(x',y))b(r,y,u(r,y))\sigma(drdy)\Big|^{2m}\Big] \\ &\leq\; \Big(\int_0^{t'}\int_{\mathbb{R}} |p_{t-r}(x,y) - p_{t'-r}(x',y)|\sigma(drdy)\Big)^{2m-1} \\ &\qquad \times \int_0^{t'}\int_{\mathbb{R}} |p_{t-r}(x,y) - p_{t'-r}(x',y)| e^{\lambda|y|}e^{-\lambda|y|}\mathbb{E}\Big[c(1+u(r,y))^{2m}\Big]\sigma(drdy) \\ &\leq\; \Big(\int_0^{t'}\int_{\mathbb{R}} |p_{t-r}(x,y) - p_{t'-r}(x',y)|\sigma(drdy)\Big)^{2m-1} \\ &\qquad\qquad \times c_T\int_0^{t'}\int_{\mathbb{R}} |p_{t-r}(x,y) - p_{t'-r}(x',y)| e^{\lambda|y|}\sigma(drdy) \\ &\leq\; c_{\lambda,T}\Big(|t-t'|^{\beta} + |x-x'|^{2\beta}\Big)^{2m} e^{\lambda|x-x'|}e^{\lambda|x|}.\end{aligned}$$

For m sufficiently large, Proposition 3.8 thus provides a $C_{tem}(\mathbb{R})$-valued continuous modification of $\Phi_2(u)$. Using Theorem 3.28, Hölder's inequality ($\frac{m-1}{m} + \frac{1}{m} = 1$), (6.15) and

Lemma 4.6 we get for all $0 \leq t \leq t' \leq T$ and $x, x' \in \mathbb{R}$:

$$\begin{aligned}
&\mathbb{E}\Big[\Big|\Phi_3(u)(t,x) - \Phi_3(u)(t',x')\Big|^{2m}\Big] \\
&= c\,\mathbb{E}\Big[\Big|\int_0^{t'}\int_{\mathbb{R}} (p_{t-r}(x,y) - p_{t'-r}(x',y))a(r,y,u(r,y))W^{\varrho}(drdy)\Big|^{2m}\Big] \\
&\leq c\,\mathbb{E}\Big[\Big|\int_0^{t'}\int_{\mathbb{R}} (p_{t-r}(x,y) - p_{t'-r}(x',y))^2 a^2(r,y,u(r,y))\varrho(drdy)\Big|^{m}\Big] \\
&\leq c\,\Big(\int_0^{t'}\int_{\mathbb{R}} (p_{t-r}(x,y) - p_{t'-r}(x',y))^2\varrho(drdy)\Big)^{m-1} \\
&\qquad \times \int_0^{t'}\int_{\mathbb{R}} (p_{t-r}(x,y) - p_{t'-r}(x',y))^2 e^{\lambda|y|}e^{-\lambda|y|}\mathbb{E}[c'(1+u(r,y))^{2m}]\varrho(drdy) \\
&\leq c_T\,\Big(\int_0^{t'}\int_{\mathbb{R}} (p_{t-r}(x,y) - p_{t'-r}(x',y))^2\varrho(drdy)\Big)^{m-1} \\
&\qquad\qquad \times \int_0^{t'}\int_{\mathbb{R}} (p_{t-r}(x,y) - p_{t'-r}(x',y))^2 e^{\lambda|y|}\varrho(drdy) \\
&\leq c_{\lambda,T}\,\big(|t-t'|^{\alpha} + |x-x'|^{2\alpha}\big)^m e^{\lambda|x-x'|}e^{\lambda|x|},
\end{aligned}$$

and so a $C_{tem}(\mathbb{R})$-valued continuous modification of $\Phi_3(u)$. By Lemma 4.9, $(P_t\eta(.) : t \geq 0)$ is $C_{tem}(\mathbb{R})$-valued continuous, too. Altogether, $\Phi(u)$ has a $C_{tem}(\mathbb{R})$-valued continuous modification $\Phi(u)'$, which, as a consequence of Lemma 4.9 and the obtained estimates, is locally Hölder-γ-continuous on $(0,\infty)\times\mathbb{R}$ for all $\gamma \in (0, \frac{\alpha}{2} \wedge \beta)$. The variables $\Phi(u)(t,x)$ are $\mathcal{F}_t$-measurable since u is $(\mathcal{F}_t)$-predictable. So, since $(\mathcal{F}_t)$ satisfies the usual conditions, the variables $\Phi(u)'(t,x)$ are also $\mathcal{F}_t$-measurable (recall Remark 3.4). Using the fact that all samples of $\Phi(u)'$ are jointly continuous, we can deduce $(\mathcal{F}_t)$-predictability of $\Phi(u)'$.

Step 2. As already mentioned, we intend a Picard-Lindelöf iteration w.r.t. the functional Φ. Set $u_0 := P_.\eta(.)$ and $u_{n+1} := \Phi(u_n)$ for all $n \geq 0$. We first show $u_n \in \mathcal{P}$ for every $n \geq 1$. In particular, we will see that

$$\sup_{n\geq 1}\|u_n\|_{\lambda,T,m} \leq c_{\lambda,T,m} < \infty \qquad \forall \lambda, T > 0,\ m \geq 1 \tag{6.17}$$

holds whenever $\eta \in C_{tem}(\mathbb{R})$. If $\eta \in C_{tem}(\mathbb{R})$, then $P_.\eta(.) \in \mathcal{P}$ is jointly continuous and it is not hard to show that $\|P_.\eta(.)\|_{\lambda,T,m}$ is finite. In particular, $u_0 = P_.\eta(.) \in \mathcal{P}$. Hence, in order to show $u_n \in \mathcal{P}$ for every $n \geq 0$, it is enough to show that $u \in \mathcal{P}$ implies $\Phi(u) \in \mathcal{P}$.

To do so, pick $u \in \mathcal{P}$. By Step 1, $\Phi(u)$ is $(\mathcal{F}_t)$-predictable. If we also had

$$\begin{aligned}
\|\Phi(u)\|_{\lambda,T,m}^{2m} \leq\ & c_{\lambda,T,m}\Big\{1 + \sup_{t\in[0,T]}\int_0^t \frac{1}{(t-r)^{1-\alpha_1/2}}\,e^{-\lambda|y|}\mathbb{E}\Big[|u(r,y)|^{2m}\Big]\varrho_2(dr) \\
&+ \sup_{t\in[0,T]}\int_0^t \frac{1}{(t-r)^{1/2-\beta_1/2}}\,e^{-\lambda|y|}\mathbb{E}\Big[|u(r,y)|^{2m}\Big]\sigma_2(dr)\Big\}
\end{aligned} \tag{6.18}$$

for all $\lambda, T > 0$ and $m \geq 1$, then $\|\Phi(u)\|_{\lambda,T,m} < \infty$, and so $\Phi(u) \in \mathcal{P}$, would follow from the finiteness of $\|u\|_{\lambda,T,m}$ and Lemma 4.4(i). We prove (6.18). Using Hölder's inequality

$(\frac{2m-1}{2m}+\frac{1}{2m}=1)$, Lemma 4.2$(i) \Rightarrow (iv)$ and (6.15) we obtain

$$
\begin{aligned}
&\|\Phi_2(u)\|^{2m}_{\lambda,T,m} \\
&\quad\le \sup_{t\le T}\sup_{x\in\mathbb{R}} e^{-\lambda|x|}\mathbb{E}\Big[\Big|\int_0^t\int_{\mathbb{R}} p_{t-r}(x,y)b(r,y,u(r,y))\sigma(drdy)\Big|^{2m}\Big] \\
&\quad\le \sup_{t\le T}\sup_{x\in\mathbb{R}} e^{-\lambda|x|}\Big(\int_0^t\int_{\mathbb{R}} p_{t-r}(x,y)\sigma(drdy)\Big)^{2m-1} \\
&\qquad\quad \times\int_0^t\int_{\mathbb{R}} e^{\lambda|y|}p_{t-r}(x,y)e^{-\lambda|y|}\mathbb{E}\Big[(1+u(r,y))^{2m}\Big]\sigma_1(r,dy)\sigma_2(dr) \\
&\quad\le c_{T,m}\Big\{1+\sup_{t\le T}\sup_{x\in\mathbb{R}}\int_0^t\Big(e^{-\lambda|x|}\int_{\mathbb{R}} e^{\lambda|y|}p_{t-r}(x,y)\sigma_1(r,dy)\Big)e^{-\lambda|y|}\mathbb{E}\Big[|u(r,y)|^{2m}\Big]\sigma_2(dr)\Big\} \\
&\quad\le c_{\lambda,T,m}\Big\{1+\sup_{t\le T}\int_0^t\frac{1}{(t-r)^{1/2-\beta_1/2}}\,e^{-\lambda|y|}\mathbb{E}\Big[|u(r,y)|^{2m}\Big]\sigma_2(dr)\Big\}.
\end{aligned}
$$

By Theorem 3.28, Hölder's inequality $(\frac{m-1}{m}+\frac{1}{m}=1)$, Lemma 4.2$(i) \Rightarrow (iv)$ and (6.15) we also obtain

$$
\begin{aligned}
&\|\Phi_3(u)\|^{2m}_{\lambda,T,m} \\
&\quad\le \sup_{t\le T}\sup_{x\in\mathbb{R}} e^{-\lambda|x|}\mathbb{E}\Big[\Big|\int_0^t\int_{\mathbb{R}} p_{t-r}(x,y)a(r,y,u(r,y))W^{\varrho}(drdy)\Big|^{2m}\Big] \\
&\quad\le c\sup_{t\le T}\sup_{x\in\mathbb{R}} e^{-\lambda|x|}\mathbb{E}\Big[\Big|\int_0^t\int_{\mathbb{R}} p^2_{t-r}(x,y)a^2(r,y,u(r,y))\varrho(drdy)\Big|^{m}\Big] \\
&\quad\le c\sup_{t\le T}\sup_{x\in\mathbb{R}} e^{-\lambda|x|}\Big(\int_0^t\int_{\mathbb{R}} p^2_{t-r}(x,y)\varrho(drdy)\Big)^{m-1} \\
&\qquad\quad \times\int_0^t\int_{\mathbb{R}} e^{\lambda|y|}p^2_{t-r}(x,y)e^{-\lambda|y|}\mathbb{E}\Big[(1+u(r,y))^{2m}\Big]\varrho_1(r,dy)\varrho_2(dr) \\
&\quad\le c_{T,m}\Big\{1+\sup_{t\le T}\sup_{x\in\mathbb{R}}\int_0^t\Big(e^{-\lambda|x|}\int_{\mathbb{R}} e^{\lambda|y|}p^2_{t-r}(x,y)\varrho_1(r,dy)\Big)e^{-\lambda|y|}\mathbb{E}\Big[|u(r,y)|^{2m}\Big]\varrho_2(dr)\Big\} \\
&\quad\le c_{\lambda,T,m}\Big\{1+\sup_{t\le T}\int_0^t\frac{1}{(t-r)^{1-\alpha_1/2}}\,e^{-\lambda|y|}\mathbb{E}\Big[|u(r,y)|^{2m}\Big]\varrho_2(dr)\Big\}.
\end{aligned}
$$

On the whole, we reach (6.18) and so $\Phi(u)\in\mathcal{P}$. In particular, $u_n\in\mathcal{P}$ for every $n\ge 0$.

The uniform estimate (6.17) is an immediate consequence of Lemma 4.11 and the following trivial conclusion of (6.18):

$$
\begin{aligned}
\|u_{n+1}\|^{2m}_{\lambda,t,m} \;\le\; & c_{\lambda,T,m}\Big\{1+\sup_{s\in[0,t]}\int_0^s\frac{1}{(s-r)^{1-\alpha_1/2}}\|u_n\|^{2m}_{\lambda,r,m}\varrho_2(dr) \\
&+\sup_{s\in[0,t]}\int_0^t\frac{1}{(s-r)^{1/2-\beta_1/2}}\|u_n\|^{2m}_{\lambda,r,m}\sigma_2(dr)\Big\}\quad \forall t\le T, n\ge 0.
\end{aligned}
$$

Step 3. We here intend to show $\lim_{n\to\infty}\|u_{n+1}-u_n\|_{\lambda,T,m}=0$ for all $\lambda,T>0$ and $m\ge 1$. By Lemma 4.11 it suffices to show that for every $\lambda,T>0$ and $m\ge 1$ there exists a constant $c_{\lambda,T,m} > 0$ (being independent of n) such that

$$\|u_{n+1} - u_n\|^{2m}_{\lambda,T,m} \leq c_{\lambda,T,m}\Big\{\sup_{t\leq T}\int_0^t \frac{1}{(t-r)^{1-\alpha_1/2}}\|u_n - u_{n-1}\|^{2m}_{\lambda,r,m}\varrho_2(dr) + \sup_{t\leq T}\int_0^t \frac{1}{(t-r)^{1/2-\beta_1/2}}\|u_n - u_{n-1}\|^{2m}_{\lambda,r,m}\sigma_2(dr)\Big\} \tag{6.19}$$

holds for all $n \geq 0$. Proceeding as in Step 2 we obtain

$$\|\Phi_2(u_n) - \Phi_2(u_{n-1})\|^{2m}_{\lambda,T,m} \leq c_{\lambda,T,m}\sup_{t\leq T}\int_0^t \frac{1}{(t-r)^{1/2-\beta_1/2}}\|u_n - u_{n-1}\|^{2m}_{\lambda,r,m}\sigma_2(dr),$$

$$\|\Phi_3(u_n) - \Phi_3(u_{n-1})\|^{2m}_{\lambda,T,m} \leq c_{\lambda,T,m}\sup_{t\leq T}\int_0^t \frac{1}{(t-r)^{1-\alpha_1/2}}\|u_n - u_{n-1}\|^{2m}_{\lambda,r,m}\varrho_2(dr)$$

where we used the Lipschitz condition (6.16) instead of (6.15). This proves (6.19).

Step 4. By Step 2 and Step 3, (u_n) is a Cauchy sequence in $\mathcal{P}$. So there exists $u_\infty \in \mathcal{P}$ such that $\lim_{n\to\infty}\|u_\infty - u_n\|_{\lambda,T,m} = 0$ for all $\lambda, T > 0$ and $m \geq 1$. Plainly,

$$\|u_\infty - \Phi(u_\infty)\|_{\lambda,T,m} \leq \|u_\infty - u_n\|_{\lambda,T,m} + \|u_n - \Phi(u_\infty)\|_{\lambda,T,m}.$$

The first summand on the r.h.s. converges to 0 as $n \to \infty$. The second summand on the r.h.s. converges to 0 as well since we obtain as before

$$\begin{aligned}\|u_n - \Phi(u_\infty)\|^{2m}_{\lambda,T,m} &\leq c_{\lambda,T,m}\Big\{\sup_{t\leq T}\int_0^t \frac{1}{(t-r)^{1-\alpha_1/2}}\|u_{n-1} - u_\infty\|^{2m}_{\lambda,r,m}\varrho_2(dr) \\ &\qquad + \sup_{t\leq T}\int_0^t \frac{1}{(t-r)^{1/2-\beta_1/2}}\|u_{n-1} - u_\infty\|^{2m}_{\lambda,r,m}\sigma_2(dr)\Big\} \\ &\leq \tilde{c}_{\lambda,T,m}\,\|u_{n-1} - u_\infty\|^{2m}_{\lambda,T,m}\end{aligned}$$

for $n \geq 1$. Hence, $\|u_\infty - \Phi(u_\infty)\|_{\lambda,T,m} = 0$. In the same way we obtain for $u := \Phi(u_\infty)$,

$$\|u - \Phi(u)\|^{2m}_{\lambda,T,m} = \|\Phi(u_\infty) - \Phi(\Phi(u_\infty))\|^{2m}_{\lambda,T,m} \leq \tilde{c}_{\lambda,T,m}\|u_\infty - \Phi(u_\infty)\|^{2m}_{\lambda,T,m} = 0.$$

Therefore, $u(t,x) = \Phi(u)(t,x)$ holds for all (t,x) from any fixed countable dense subset of $[0,\infty)\times\mathbb{R}$, $\mathbb{P}$-almost surely. Also, by Step 1, u may be assumed to be $C_{tem}(\mathbb{R})$-valued continuous and so $u(t,x) = \Phi(u)(t,x)$ even holds for all (t,x), $\mathbb{P}$-almost surely. Consequently, u is a solution of SIE (6.7). Step 1 also gives the desired local Hölder-continuity.

Step 5. It remains to show strong uniqueness of solutions. Let u, u' be two solutions to SPDE (5.9) and so to $u = \Phi(u)$. Fix some $\lambda > 0$. For every $K > 0$ define the stopping time

$$\tau_K := \inf\{t > 0 : \sup_{x\in\mathbb{R}}|u(t,x)|e^{-(\lambda/2)|x|} \geq K \text{ or } \sup_{x\in\mathbb{R}}|u'(t,x)|e^{-(\lambda/2)|x|} \geq K\}$$

as well as $u_K(t,.) := \mathbf{1}_{t<\tau_K} u(t,.)$ and $u'_K(t,.) := \mathbf{1}_{t<\tau_K} u'(t,.)$. As in Step 3 we get

$$\begin{aligned}
\|u_K - u'_K\|^2_{\lambda,t,1} &= \|\Phi(u_K) - \Phi(u'_K)\|^2_{\lambda,t,1} \\
&\leq c_{\lambda,T,1}\Big\{ \sup_{s\leq t} \int_0^{s\wedge\tau_K} \frac{1}{(s-r)^{1-\alpha_1/2}} \|u_K - u'_K\|^2_{\lambda,r,1} \varrho_2(dr) \\
&\qquad + \sup_{s\leq t} \int_0^{s\wedge\tau_K} \frac{1}{(s-r)^{1/2-\beta_1/2}} \|u_K - u'_K\|^2_{\lambda,r,1} \sigma_2(dr) \Big\}
\end{aligned}$$

for all $t \leq T$, and every $T > 0$. Since $|u_K(t,x) - u'_K(t,x)|^2 \leq 4K^2 e^{\lambda|x|}$ for all (t,x), Lemma 4.12 gives $\|u_K - u'_K\|_{\lambda,t,1} = 0$ for all $t \geq 0$. In particular, $u_K(t,x) = u'_K(t,x)$ holds for all (t,x) from any fixed countable dense subset of $[0,\infty) \times \mathbb{R}$, $\mathbb{P}$-almost surely. However, u and u' are $C_{tem}(\mathbb{R})$-valued continuous, i.e. $\tau_K \to \infty$ as $K \to \infty$ $\mathbb{P}$-almost surely. Thus, $u(t,x) = u'(t,x)$ holds for all (t,x) from $[0,\infty) \times \mathbb{R}$, $\mathbb{P}$-almost surely. This completes the proof of Theorem 6.8. □

6.4 Non-negativity of solutions

In the previous section we have seen that SPDE (5.9) possesses strongly unique strong solutions whenever the coefficients are Lipschitz continuous and grow at most linearly. Under slightly stronger assumptions we obtain non-negativity of the solutions.

Theorem 6.9 [NON-NEGATIVITY] *Let a and b be continuous and $\kappa > 0$. Assume for every $T > 0$ there exist finite constants $c_T, L_T > 0$ such that for all $t \leq T$ and $x, x', u, u' \in \mathbb{R}$*

$$|a(t,x,u)| + |b(t,x,u)| \leq c_T(1 + |u|),$$

$$|a(t,x,u) - a(t,x',u')| + |b(t,x,u) - b(t,x',u')| \leq L_T(|x - x'|^\kappa + |u - u'|)$$

hold. Also, let $\varrho(dtdx)$ and $\sigma(dtdx)$ satisfy condition (A) *and* (B)*, respectively. If moreover $a(t,x,0) = 0$, $b(t,x,0) \geq 0$ ($\forall t \geq 0, x \in \mathbb{R}$) and $\eta \in C^+_{tem}(\mathbb{R})$, then the unique solution from Theorem 6.8 is $\mathbb{P}$-almost surely non-negative.*

Shiga showed non-negativity of solutions to SPDE (5.9) with $\varrho(dtdx) = \sigma(dtdx) = dtdx$, see the appendix of [Shi94]. In order to prove Theorem 6.9 we adapt his arguments. For all $t \geq 0$, $x \in \mathbb{R}$ and $\epsilon > 0$ define:

$$\begin{aligned}
\Delta_\epsilon &:= \epsilon^{-1}(P_\epsilon - \mathbb{I}) \\
P^\epsilon_t &:= e^{t\Delta_\epsilon} = e^{-t/\epsilon} e^{t/\epsilon P_\epsilon} = e^{-t/\epsilon} \sum_{n=0}^\infty \frac{(t/\epsilon)^n}{n!} P_{n\epsilon} = e^{-t/\epsilon}\mathbb{I} + Q^\epsilon_t \\
Q^\epsilon_t f &:= \int_{\mathbb{R}} q^\epsilon_t(.,y) f(y) dy, \qquad q^\epsilon_t(x,y) := e^{-t/\epsilon} \sum_{n=1}^\infty \frac{(t/\epsilon)^n}{n!} p_{n\epsilon}(x,y).
\end{aligned}$$

Before proving Theorem 6.9 we present a number of lemmas.

Lemma 6.10 *For all $\epsilon > 0$, $(P^\epsilon_t) \equiv (P^\epsilon_t)_{t\geq 0}$ provides a strongly continuous contraction semigroup of linear operators on $(C_0(\mathbb{R}), \|.\|_\infty)$.*

Proof P_ϵ and $\mathbb{I}$ are linear bounded operators on $C_0(\mathbb{R})$ and so is Δ_ϵ. Thus, $e^{s\Delta_\epsilon}e^{t\Delta_\epsilon} = e^{(s+t)\Delta_\epsilon}$ holds on $C_0(\mathbb{R})$, i.e. (P_t) is a semigroup on $C_0(\mathbb{R})$. Also, P_t^ϵ is easily seen to be linear for every $t \geq 0$. The strong continuity follows from

$$\begin{aligned}\|P_t^\epsilon f - f\|_\infty &= \|\sum_{n=0}^\infty \frac{t^n}{n!}\Delta_\epsilon^n f - f\|_\infty = \|\sum_{n=1}^\infty \frac{t^n}{n!}\Delta_\epsilon^n f\|_\infty \\ &\leq \sum_{n=1}^\infty \frac{t^n}{n!}\|\Delta_\epsilon^n\|_\infty \|f\|_\infty \leq \sum_{n=1}^\infty \frac{t^n}{n!}\|\Delta_\epsilon\|_\infty^n \|f\|_\infty \leq \left(e^{t\|\Delta_\epsilon\|_\infty} - 1\right)\|f\|_\infty\end{aligned}$$

since the latter estimate tends to zero as $t \downarrow 0$, for every $f \in C_0(\mathbb{R})$. As P_t is contractive for every $t \geq 0$, we further obtain for every $f \in C_0(\mathbb{R})$:

$$\|P_t^\epsilon f\|_\infty = \left\|e^{-t/\epsilon}\sum_{n=0}^\infty \frac{(t/\epsilon)^n}{n!}P_{n\epsilon}f\right\|_\infty \leq e^{-t/\epsilon}\sum_{n=0}^\infty \frac{(t/\epsilon)^n}{n!}\|P_{n\epsilon}f\|_\infty \leq \|f\|_\infty.$$

That is, P_t^ϵ is contractive as well (for every $t \geq 0$). □

Lemma 6.11 *For all $\epsilon > 0$, Δ_ϵ (defined on $C_0(\mathbb{R})$) is the generator of (P_t^ϵ).*

Proof For every $f \in C_0(\mathbb{R})$ and $\epsilon > 0$ we obtain

$$\begin{aligned}&\|t^{-1}(P_t^\epsilon - \mathbb{I})f - \Delta_\epsilon f\|_\infty \\ &= \left\|t^{-1}e^{-t/\epsilon}\sum_{n=0}^\infty \frac{(t/\epsilon)^n}{n!}(P_{n\epsilon}f - f) - \Delta_\epsilon f\right\|_\infty \\ &= \left\|t^{-1}e^{-t/\epsilon}\sum_{n=1}^\infty \frac{(t/\epsilon)^n}{n!}(P_{n\epsilon}f - f) - \Delta_\epsilon f\right\|_\infty \\ &= \left\|\frac{1}{\epsilon}e^{-t/\epsilon}\sum_{n=1}^\infty \frac{(t/\epsilon)^{n-1}}{n!}(P_{n\epsilon}f - f) - \frac{1}{\epsilon}(P_\epsilon f - f)\right\|_\infty \\ &\leq \left\|\frac{1}{\epsilon}e^{-t/\epsilon}(P_\epsilon f - f) - \frac{1}{\epsilon}(P_\epsilon f - f)\right\|_\infty + \left\|\frac{1}{\epsilon}e^{-t/\epsilon}\sum_{n=2}^\infty \frac{(t/\epsilon)^{n-1}}{n!}(P_{n\epsilon}f - f)\right\|_\infty \\ &\leq |e^{-t/\epsilon} - 1|\left\|\frac{1}{\epsilon}(P_\epsilon f - f)\right\|_\infty + \frac{t}{\epsilon^2}e^{-t/\epsilon}\sum_{n=2}^\infty \frac{(t/\epsilon)^{n-2}}{(n-2)!}\,2\,\|f\|_\infty \\ &\leq c_{\epsilon,f}(|e^{-t/\epsilon} - 1| + t) \quad \to 0 \quad \text{as } t \downarrow 0\end{aligned}$$

where we used $\|P_s\|_\infty = 1$ ($\forall s \geq 0$) and $\|\mathbb{I}\|_\infty = 1$. □

Lemma 6.12 *For every $f \in C_0(\mathbb{R})$ and $t \geq 0$, $\lim_{\epsilon\downarrow 0}\|P_t^\epsilon f - P_t f\|_\infty = 0$ holds.*

Proof For every $f \in C_0(\mathbb{R})$ and $\epsilon, \epsilon' > 0$ we have (w.r.t. $\|.\|_\infty$):

$$\frac{d}{ds} e^{s(\Delta_\epsilon - \Delta_{\epsilon'})} f = e^{s(\Delta_\epsilon - \Delta_{\epsilon'})}(\Delta_\epsilon - \Delta_{\epsilon'})f.$$

Integrating over $[0,t]$ leads to

$$e^{t(\Delta_\epsilon - \Delta_{\epsilon'})} f - f = \int_0^t e^{s(\Delta_\epsilon - \Delta_{\epsilon'})}(\Delta_\epsilon - \Delta_{\epsilon'}) f ds.$$

Applying the operator $e^{t\Delta_{\epsilon'}}$ on both sides yields

$$e^{t\Delta_\epsilon} f - e^{t\Delta_{\epsilon'}} f = \int_0^t e^{s\Delta_\epsilon} e^{(t-s)\Delta_{\epsilon'}}(\Delta_\epsilon - \Delta_{\epsilon'}) f ds$$

and so, since $\|e^{r\Delta_\epsilon}\|_\infty \leq 1$, for every $t \geq 0$:

$$\begin{aligned} \|P_t^\epsilon f - P_t^{\epsilon'} f\|_\infty &= \|e^{t\Delta_\epsilon} f - e^{t\Delta_{\epsilon'}} f\|_\infty \\ &\leq \int_0^t \|e^{s\Delta_\epsilon}\|_\infty \|e^{(t-s)\Delta_{\epsilon'}}\|_\infty \|\Delta_\epsilon f - \Delta_{\epsilon'} f\|_\infty ds \\ &\leq \int_0^t \|\Delta_\epsilon f - \Delta_{\epsilon'} f\|_\infty ds \; = \; t \, \|\Delta_\epsilon f - \Delta_{\epsilon'} f\|_\infty. \end{aligned}$$

Hence, $\lim_{\epsilon,\epsilon' \downarrow 0} \|P_t^\epsilon f - P_t^{\epsilon'} f\|_\infty = 0$ uniformly in t on compacts. So we can define

$$\tilde{P}_t f := \|.\|_\infty\text{-}\lim_{\epsilon \downarrow 0} P_t^\epsilon f, \qquad f \in C_0(\mathbb{R}) \tag{6.20}$$

where the convergence is uniform in t on compacts. It is not hard to verify that $(\tilde{P}_t)$ provides a strongly continuous semigroup. Also, its generator on $C_0(\mathbb{R})$ is $\frac{1}{2}\Delta$ since

$$\lim_{t \downarrow 0} \frac{1}{t}(\tilde{P}_t f - f) = \lim_{t \downarrow 0} \lim_{\epsilon \downarrow 0} \frac{1}{t}(P_t^\epsilon f - f) = \lim_{\epsilon \downarrow 0} \lim_{t \downarrow 0} \frac{1}{t}(P_t^\epsilon f - f) = \lim_{\epsilon \downarrow 0} \Delta_\epsilon f = \frac{1}{2}\Delta f$$

holds w.r.t. $\|.\|_\infty$ for every $f \in C_0(\mathbb{R})$. However, two strongly continuous semigroups with the same generator coincide (cf. [Wer00], Korollar VII.4.8), i.e. $(\tilde{P}_t) \equiv (P_t)$. So the claim follows from (6.20). $\square$

Lemma 6.13 *There exists a finite constant $c > 0$ such that $e^{-h} \sum_{n=1}^\infty \frac{h^n}{n!} \frac{h^\gamma}{n^\gamma} \leq c$ holds for all $\gamma \in [0,1]$ and $h \geq 0$.*

Proof For $h \in [0,1]$ the claim is trivial. Suppose $h > 1$. Define $[h]$ to be the unique integer satisfying $[h] \leq h < [h]+1$. We plainly have $(\frac{h}{n})^\gamma \leq \frac{h}{n}$ if $\frac{h}{n} \geq 1$ (i.e. $n \leq [h] \leq h$) and $(\frac{h}{n})^\gamma \leq 1$ if $\frac{h}{n} < 1$ (i.e. $n \geq [h]+1 > h$). Therefore,

$$\begin{aligned} e^{-h} \sum_{n=1}^\infty \frac{h^n}{n!} \frac{h^\gamma}{n^\gamma} \; &\leq \; e^{-h} \Big(\sum_{n=1}^{[h]} \frac{h^n}{n!} \frac{h}{n} + \sum_{[h]+1}^\infty \frac{h^n}{n!} \Big) \\ &\leq \; e^{-h} \Big(\sum_{n=1}^{[h]} \frac{h^{n+1}}{(n+1)!} \frac{n+1}{n} + \sum_{[h]+1}^\infty \frac{h^n}{n!} \Big) \; \leq \; e^{-h} \Big(\sum_{n=2}^\infty \frac{h^n}{n!} 2 + \sum_{n=1}^\infty \frac{h^n}{n!} \Big) \; \leq \; c. \end{aligned}$$

$\square$

Lemma 6.14 *Let $i \in \{1,2\}$. For every $\lambda \geq 0$ and $R > 0$ there exists a finite constant $c_{\lambda,R} > 0$ such that for all $x \in \mathbb{R}$, $t > 0$ and $\epsilon \in (0,1]$:*

$$\sup_{r\in[0,R]} \int_{\mathbb{R}} \int_{\mathbb{R}} q_t^\epsilon(x,z)^i p_\epsilon(z,y) e^{\lambda|\xi|} dz \varrho_1(r,dy) \;\leq\; c_{\lambda,R} \frac{1}{t^{i/2-\alpha_1/2}} e^{\lambda|x|}, \quad \xi \in \{y,z\},$$

$$\sup_{r\in[0,R]} \int_{\mathbb{R}} \int_{\mathbb{R}} q_t^\epsilon(x,z) p_\epsilon(z,y) e^{\lambda|\xi|} dz \sigma_1(r,dy) \;\leq\; c_{\lambda,R} \frac{1}{t^{1/2-\beta_1/2}} e^{\lambda|x|}, \quad \xi \in \{y,z\}.$$

Proof We only prove the first line for the case $i = 2$ and $\xi = z$. The other cases can be shown analogously. Using Lemma 6.13, Hölder's inequality, Lemma 4.2 and Remark 4.3 we obtain the following estimate

$$\begin{aligned}
&\int_{\mathbb{R}} \int_{\mathbb{R}} q_t^\epsilon(x,z)^2 p_\epsilon(z,y) e^{\lambda|z|} dz \varrho_1(r,dy) \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-2t/\epsilon} \Big(\sum_{n=1}^\infty \frac{(t/\epsilon)^n}{n!} p_{n\epsilon}(x,z) \Big)^2 p_\epsilon(y,z) e^{\lambda|z|} dz \varrho_1(r,dy) \\
&\leq e^{-t/\epsilon} \sum_{n=1}^\infty \frac{(t/\epsilon)^n}{n!} \int_{\mathbb{R}} \int_{\mathbb{R}} p_{n\epsilon}(x,z) p_\epsilon(z,y) e^{\lambda|z|} dz \varrho_1(r,dy) \frac{e^{-t/\epsilon}}{t^{1/2}} \sum_{n=1}^\infty \frac{(\frac{t}{\epsilon})^n}{n!} \frac{(\frac{t}{\epsilon})^{1/2}}{n^{1/2}} \\
&\leq e^{-t/\epsilon} \sum_{n=1}^\infty \frac{(t/\epsilon)^n}{n!} \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} p_{n\epsilon}(x,z) p_\epsilon(z,y) dz \Big)^{1/2} \\
&\qquad \times \Big(\int_{\mathbb{R}} p_{n\epsilon}(x,z) p_\epsilon(z,y) e^{2\lambda|z|} dz \Big)^{1/2} \varrho_1(r,dy) \; \frac{c}{t^{1/2}} \\
&\leq c \frac{1}{t^{1/2}} \; e^{-t/\epsilon} \sum_{n=1}^\infty \frac{(t/\epsilon)^n}{n!} \int_{\mathbb{R}} p_{n\epsilon+\epsilon}(x,y)^{1/2} \frac{1}{(2\pi n\epsilon)^{1/4}} e^{\lambda|y|} \varrho_1(r,dy) \\
&\leq c' \frac{1}{t^{1/2}} \; e^{-t/\epsilon} \sum_{n=1}^\infty \frac{(t/\epsilon)^n}{n!} \; c_{\lambda,R} \frac{1}{(n\epsilon+\epsilon)^{1/4-\alpha_1/2}} e^{\lambda|x|} \frac{1}{(2\pi n\epsilon)^{1/4}} \\
&\leq c'_{\lambda,R} \frac{1}{t^{1-\alpha_1/2}} \Big(e^{-t/\epsilon} \sum_{n=1}^\infty \frac{(\frac{t}{\epsilon})^n}{n!} \frac{(\frac{t}{\epsilon})^{\frac{1}{2}-\frac{\alpha_1}{2}}}{n^{\frac{1}{2}-\frac{\alpha_1}{2}}} \Big) \Big(\frac{n\epsilon}{n\epsilon+\epsilon} \Big)^{\frac{1}{4}-\frac{\alpha_1}{2}} e^{\lambda|x|} \;\leq\; c''_{\lambda,R} \frac{e^{\lambda|x|}}{t^{1-\alpha_1/2}}
\end{aligned}$$

for all $r \in [0,R]$. □

Lemma 6.15 *Let $\gamma > 0$ and $\lambda \geq 0$. Then we have for all $\epsilon \in (0,1]$:*

$$\sup_{x\in\mathbb{R}} \int_{\mathbb{R}} p_\epsilon(x,y) |x-y|^\gamma e^{\lambda|x-y|} dy \;\leq\; c_\lambda \, \epsilon^{\gamma/4}.$$

Proof With help of Hölder's inequality we obtain

$$\begin{aligned}
&\int_{\mathbb{R}} p_\epsilon(x,y) |x-y|^\gamma e^{\lambda|x-y|} dy \\
&\leq \frac{1}{\sqrt{2\pi\epsilon}} \Big(\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{2\epsilon}} |x-y|^{2\gamma} dy \Big)^{1/2} \Big(\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{2\epsilon}} e^{\lambda|x-y|} dy \Big)^{1/2} \\
&\leq \frac{1}{\sqrt{2\pi\epsilon}} \Big(c \, \epsilon^{\gamma/2+1/2} \Big)^{1/2} \Big(c_\lambda \, \epsilon^{1/2} \Big)^{1/2} \;\leq\; \tilde{c}_\lambda \, \epsilon^{\gamma/4}.
\end{aligned}$$

$\square$

Lemma 6.16 *For every $\delta > 0$ and $T > 0$ we have:*

$$\lim_{\epsilon\downarrow 0} \sup_{x,y\in\mathbb{R}} \sup_{t\le T} t^{1/2+\delta} \left| \int_{\mathbb{R}} q_t^{\epsilon}(x,z)p_{\epsilon}(y,z)dz - p_t(x,y)\right| = 0.$$

The technical proof of Lemma 6.16 will be omitted.

Proof (of Theorem 6.9) For every $\epsilon > 0$ define the measure $\sigma_x^{\epsilon}(dt) := \int_{\mathbb{R}} p_{\epsilon}(x,y)\sigma(dtdy)$ and the time white noise $W_x^{\epsilon}(dt) := \int_{\mathbb{R}} p_{\epsilon}(x,y)W^{\varrho}(dtdy)$ (formally). Note that, for every $\epsilon > 0$, $\sigma_x^{\epsilon}(t) := \int_0^t \sigma_x^{\epsilon}(dr)$ is a (deterministic) non-negative non-decreasing function on $[0,\infty)$ and $W_x^{\epsilon}(t) = \int_0^t W_x^{\epsilon}(dr) := \int_0^t \int_{\mathbb{R}} p_{\epsilon}(x,y)W_x^{\varrho}(drdy)$ is a continuous square-integrable martingale with quadratic variation process $\langle W_x^{\epsilon}\rangle(t) = \int_0^t \int_{\mathbb{R}} p_{\epsilon}^2(x,y)\varrho(drdy)$.

The strategy of the proof is as follows. As the first step (Step 1) we shall prove that, for every fixed $\epsilon > 0$, the following equation family with index $x \in \mathbb{R}$

$$\begin{aligned} u_{\epsilon}(t,x) &= \eta(x) \\ &\quad + \int_0^t \Delta_{\epsilon} u_{\epsilon}(r,x)dr + \int_0^t b(r,x,u_{\epsilon}(r,x))\sigma_x^{\epsilon}(dr) + \int_0^t a(r,x,u_{\epsilon}(r,x))W_x^{\epsilon}(dr) \end{aligned} \tag{6.21}$$

has a unique $C_{tem}(\mathbb{R})$-valued continuous solution u_{ϵ} (where the last term is an Itô-integral against the martingale W_x^{ϵ}). Then we show that u_{ϵ} is non-negative (Step 2). In Step 3 we approximate the unique solution u of SPDE (5.9) by u_{ϵ} ($\epsilon \downarrow 0$) whereby the desired non-negativity of u will follow. The approximation of u by u_{ϵ} is not surprising since the equation family (6.21) is equivalent to the following mollified version of SPDE (5.9):

$$\begin{aligned} \langle u_{\epsilon}(t,.),\psi\rangle &= \langle \eta,\psi\rangle + \int_0^t \langle u_{\epsilon}(r,.),\Delta_{\epsilon}\psi\rangle dr \\ &\quad + \int_0^t \int_{\mathbb{R}}\int_{\mathbb{R}} b(r,z,u_{\epsilon}(r,z))\psi(z)p_{\epsilon}(y,z)dz\sigma(drdy) \\ &\quad + \int_0^t \int_{\mathbb{R}}\int_{\mathbb{R}} a(r,z,u_{\epsilon}(r,z))\psi(z)p_{\epsilon}(y,z)dzW^{\varrho}(drdy) \end{aligned} \tag{6.22}$$

for every $t \ge 0$ and $\psi \in C_c^{\infty}(\mathbb{R})$. The key for the proof of the equivalence is Lemma 5.18 (with $p_{\epsilon}(.,z)\cdot W^{\varrho}$ in place of M) and the fact that $\langle P_{\epsilon}\phi,\psi\rangle = \langle \phi, P_{\epsilon}\psi\rangle$ (and so $\langle \Delta_{\epsilon}\phi,\psi\rangle = \langle \phi,\Delta_{\epsilon}\psi\rangle$) holds for all $\phi \in C_{tem}(\mathbb{R})$ and $\psi \in C_c^{\infty}(\mathbb{R})$. We omit the details.

Step 1. We here establish a unique solution to (6.21). The crucial point is that (6.22), and so (6.21), is equivalent to the following mollified version of SIE (6.7):

$$\begin{aligned} u_{\epsilon}(t,x) &= P_t^{\epsilon}\eta(x) + \int_0^t \int_{\mathbb{R}} e^{-(t-r)/\epsilon} b(r,x,u_{\epsilon}(r,x))p_{\epsilon}(x,y)\sigma(drdy) \\ &\quad + \int_0^t \int_{\mathbb{R}}\int_{\mathbb{R}} q_{t-r}^{\epsilon}(x,z)b(r,z,u_{\epsilon}(r,z))p_{\epsilon}(y,z)dz\sigma(drdy) \\ &\quad + \int_0^t \int_{\mathbb{R}} e^{-(t-r)/\epsilon} a(r,x,u_{\epsilon}(r,x))p_{\epsilon}(x,y)W^{\varrho}(drdy) \\ &\quad + \int_0^t \int_{\mathbb{R}}\int_{\mathbb{R}} q_{t-r}^{\epsilon}(x,z)a(r,z,u_{\epsilon}(r,z))p_{\epsilon}(y,z)dzW^{\varrho}(drdy). \end{aligned} \tag{6.23}$$

The proof of the equivalence works analogously to the proof of Proposition 6.7 (recall that Δ_ϵ was the generator of P^ϵ_t). Then mimic the proof of Theorem 6.8 to obtain a unique $C_{tem}(\mathbb{R})$-valued continuous solution to SIE (6.23). This time one has to choose $\Phi^\epsilon(u) := P^\epsilon_{\cdot}\eta(.) + \Phi^\epsilon_{2,1}(u) + \Phi^\epsilon_{2,2}(u) + \Phi^\epsilon_{3,1}(u) + \Phi^\epsilon_{3,2}(u) :=$ r.h.s. of (6.23). Note that the essential technical tools are Lemmas 4.2 and 4.4 as before, as well as Lemma 6.14 and analogues of Lemma 4.6 and Lemma 4.7. In particular, one obtains $\sup_{\epsilon\in(0,1]} \|u_\epsilon\|_{\lambda,T,m} < \infty$ for all $\lambda, T > 0$ and $m \geq 1$.

Step 2. Let us turn to the non-negativity of solutions to (6.21). Choose a sequence $(x_n) \subset (-\infty, 0)$ in such a manner that $x_0 = -1$, $x_n \uparrow 0$ and $\int_{x_{n-1}}^{x_n} x^{-2}dx = n$. For every $n \geq 1$ pick a real-valued continuous function g_n on $\mathbb{R}$ such that $\mathrm{supp}(g_n) \subset (x_{n-1}, x_n)$, $0 \leq g_n(x) \leq \frac{2x^{-2}}{n}$ and $\int_{x_{n-1}}^{x_n} g_n(x)dx = 1$. Set

$$f_n(x) := \begin{cases} \int_x^0 \int_y^0 g_n(z)dzdy & , \quad x < 0 \\ 0 & , \quad x \geq 0 \end{cases}.$$

The functions f_n are obviously in $C^2(\mathbb{R})$. We have

$$f_n'(x) = \begin{cases} -\int_x^0 g_n(y)dy & , \quad x < 0 \\ 0 & , \quad x \geq 0 \end{cases} \qquad f_n''(x) = \begin{cases} g_n(x) & , \quad x < 0 \\ 0 & , \quad x \geq 0 \end{cases}.$$

One should interpret f_n, f_n' and f_n'' as approximations of $f(x) := -\min\{0, x\}$, "$f'(x)$" $= -\mathbf{1}_{(-\infty,0]}(x)$ and "$f''(x)$" $= \delta_0(x)$, respectively. In particular, we have $0 \leq f(x) - f_n(x) \leq -x_{n-1} \downarrow 0$ as $n \to \infty$ and $-f_n'(x) \in [0,1]$ for all $x \in \mathbb{R}$. Set $\tilde{u}_\epsilon(t,x) = e^{-|x|}u_\epsilon(t,x)$. Then $\tilde{u}_\epsilon(t,x) = e^{-|x|}[\eta(x) + A^\epsilon_x(t) + M^\epsilon_x(t)] := e^{-|x|}$[r.h.s. of (6.21)] is a semimartingale for every $x \in \mathbb{R}$ and Itô's formula (Theorem 5.6) yields

$$\begin{aligned} f_n(\tilde{u}_\epsilon(t,x)) &= f_n(e^{-|x|}\eta(x)) + \int_0^t f_n^{'}(\tilde{u}_\epsilon(r,x))e^{-|x|}dA^\epsilon_x(r) \\ &\quad + \int_0^t f_n^{'}(\tilde{u}_\epsilon(r,x))e^{-|x|}dM^\epsilon_x(r) + \frac{1}{2}\int_0^t f_n^{''}(\tilde{u}_\epsilon(r,x))e^{-2|x|}d\langle M^\epsilon_x\rangle(r). \end{aligned}$$

Taking expectation and using $f_n'' = g_n$, $f_n'(u) = 0$ for $u \geq 0$, $-b(r,y,u) \leq L_T|u|$ for $u \in \mathbb{R}$, $|a(r,y,u)| \leq L_T|u|$ for $u \in \mathbb{R}$, $-f_n'(u) \in [0,1]$ for $u \in \mathbb{R}$, Lemma 4.2 and $-u \leq f(u)$ for $u \in \mathbb{R}$, we obtain for every $T > 0$:

$$\begin{aligned} &\mathbb{E}\Big[f_n(\tilde{u}_\epsilon(t,x))\Big] \\ &= \mathbb{E}\Big[\int_0^t f_n^{'}(\tilde{u}_\epsilon(r,x))e^{-|x|}dA^\epsilon_x(r)\Big] + \frac{1}{2}\mathbb{E}\Big[\int_0^t f_n^{''}(\tilde{u}_\epsilon(r,x))e^{-2|x|}d\langle M^\epsilon_x\rangle(r)\Big] \\ &= \mathbb{E}\Big[\int_0^t f_n'(\tilde{u}_\epsilon(r,x))(\Delta_\epsilon u_\epsilon(r,x))e^{-|x|}dr\Big] \\ &\quad + \mathbb{E}\Big[\int_0^t \int_{\mathbb{R}} f_n'(\tilde{u}_\epsilon(r,x))b(r,x,u_\epsilon(r,x))p_\epsilon(x,y)e^{-|x|}\sigma(drdy)\Big] \\ &\quad + \frac{1}{2}\mathbb{E}\Big[\int_0^t \int_{\mathbb{R}} f_n''(\tilde{u}_\epsilon(r,x))a^2(r,x,u_\epsilon(r,x))p^2_\epsilon(x,y)e^{-2|x|}\varrho(drdy)\Big] \end{aligned}$$

$$
\begin{aligned}
= \quad & \frac{1}{\epsilon}\mathbb{E}\Big[\int_0^t f_n'(\tilde{u}_\epsilon(r,x))\int_{\mathbb{R}} p_\epsilon(x,y)\tilde{u}_\epsilon(r,y)e^{(|y|-|x|)}dydr \\
& \qquad\qquad\qquad\qquad\qquad - \int_0^t f_n'(\tilde{u}_\epsilon(r,x))\tilde{u}_\epsilon(r,x)dr\Big] \\
& +\mathbb{E}\Big[\int_0^t\int_{\mathbb{R}}\Big(-f_n'(\tilde{u}_\epsilon(r,x))\Big)\Big(-b(r,x,u_\epsilon(r,x))\Big)e^{-|x|}p_\epsilon(x,y)\sigma(drdy)\Big] \\
& +\frac{1}{2}\mathbb{E}\Big[\int_0^t\int_{\mathbb{R}} g_n(\tilde{u}_\epsilon(r,x))a^2(r,x,u_\epsilon(r,x))p_\epsilon^2(x,y)e^{-2|x|}\varrho(drdy)\Big] \\
\leq \quad & \frac{1}{\epsilon}\mathbb{E}\Big[\int_0^t f_n'(\tilde{u}_\epsilon(r,x))\int_{\mathbb{R}} p_\epsilon(x,y)\tilde{u}_\epsilon(r,y)e^{|y|}dydr\Big]e^{-|x|} \\
& +\mathbb{E}\Big[\int_0^t\int_{\mathbb{R}}\Big(-f_n'(\tilde{u}_\epsilon(r,x))\ L_T|\tilde{u}_\epsilon(r,x)|\Big)p_\epsilon(x,y)\sigma(drdy)\Big] \\
& +\frac{1}{2}\mathbb{E}\Big[\int_0^t\int_{\mathbb{R}}\frac{2|\tilde{u}_\epsilon(r,x)|^{-2}}{n}L_T^2|\tilde{u}_\epsilon(r,x)|^2p_\epsilon^2(x,y)\varrho(drdy)\Big] \\
\leq \quad & \frac{1}{\epsilon}\mathbb{E}\Big[\int_0^t\Big(-f_n'(\tilde{u}_\epsilon(r,x))\Big)\int_{\mathbb{R}} p_\epsilon(x,y)e^{|y|}(-\tilde{u}_\epsilon(r,y))dydr\Big]e^{-|x|} \\
& +c_{\epsilon,T}\mathbb{E}\Big[\int_0^t\Big(-f_n'(\tilde{u}_\epsilon(r,x))\Big)\Big(-\tilde{u}_\epsilon(r,x)\Big)\sigma_2(dr)\Big] \\
& +\frac{L_T^2}{n}\int_0^t\int_{\mathbb{R}} p_\epsilon^2(x,y)\varrho(drdy) \\
\leq \quad & c_\epsilon\int_0^t\sup_{y\in\mathbb{R}}\mathbb{E}\Big[f(\tilde{u}_\epsilon(r,y))\Big]dr + c_{\epsilon,T}\int_0^t\sup_{x\in\mathbb{R}}\mathbb{E}\Big[f(\tilde{u}_\epsilon(r,x))\Big]\sigma_2(dr)+\frac{\bar{c}_{\epsilon,T}}{n}
\end{aligned}
$$

for all $t \leq T$ and $x \in \mathbb{R}$. Letting $n \to \infty$ we infer by the dominated convergence theorem and the convergence of f_n to f that

$$
\Big\|\mathbb{E}\Big[f(\tilde{u}_\epsilon(t,.))\Big]\Big\|_\infty \leq c_\epsilon\int_0^t\Big\|\mathbb{E}\Big[f(\tilde{u}_\epsilon(r,.))\Big]\Big\|_\infty dr + c_{\epsilon,T}\int_0^t\Big\|\mathbb{E}\Big[f(\tilde{u}_\epsilon(r,.))\Big]\Big\|_\infty\sigma_2(dr)
$$

holds for all $t \leq T$, for every $T > 0$. Therefore, we deduce by Lemma 4.11 that $\sup_{t\leq T}\|\mathbb{E}[f(\tilde{u}_\epsilon(t,.))]\|_\infty = 0$ holds for each $T > 0$. Since $f \geq 0$, we conclude $f(\tilde{u}_\epsilon(t,x)) = 0$ $\mathbb{P}$-almost surely, for all (t,x). Hence, $f(\tilde{u}_\epsilon(t,x)) = 0$ holds for all rational couples (t,x), $\mathbb{P}$-almost surely. The joint continuity of $\tilde{u}_\epsilon$ finally implies $\tilde{u}_\epsilon(t,x) \geq 0$ (and so $u_\epsilon(t,x) \geq 0$) for all (t,x), $\mathbb{P}$-almost surely.

Step 3. We now approximate u by the u_ϵ. Plainly,

$$
\begin{aligned}
& e^{-\lambda|x|}\mathbb{E}\Big[|u_\epsilon(t,x)-u(t,x)|^2\Big] \ \leq\ e^{-\lambda|x|}2\Big\{\ |P_t^\epsilon\eta(x)-P_t\eta(x)|^2 \\
& +\mathbb{E}\Big[\Big|\textstyle\int_0^t\int e^{-\frac{t-r}{\epsilon}}b(r,x,u_\epsilon(r,x))p_\epsilon(x,y)\sigma(drdy)\Big|^2\Big] \\
& +\mathbb{E}\Big[\Big|\textstyle\int_0^t\int\int q_{t-r}^\epsilon(x,z)\Big(b(r,z,u_\epsilon(r,z))-b(r,z,u(r,z))\Big)p_\epsilon(y,z)dz\sigma(drdy)\Big|^2\Big] \\
& +\mathbb{E}\Big[\Big|\textstyle\int_0^t\int\int q_{t-r}^\epsilon(x,z)\Big(b(r,z,u(r,z))-b(r,y,u(r,y))\Big)p_\epsilon(y,z)dz\sigma(drdy)\Big|^2\Big]
\end{aligned}
$$

$$+\mathbb{E}\Big[\Big|\int_0^t\int\Big(\int_{\mathbb{R}}q^{\epsilon}_{t-r}(x,z)p_\epsilon(y,z)dz-p_{t-r}(x,y)\Big)b(r,y,u(r,y))\sigma(drdy)\Big|^2\Big]$$

$$+\mathbb{E}\Big[\Big|\int_0^t\int e^{-\frac{t-r}{\epsilon}}a(r,x,u_\epsilon(r,x))p_\epsilon(x,y)W^{\varrho}(drdy)\Big|^2\Big]$$

$$+\mathbb{E}\Big[\Big|\int_0^t\int\int q^{\epsilon}_{t-r}(x,z)\Big(a(r,z,u_\epsilon(r,z))-a(r,z,u(r,z))\Big)p_\epsilon(y,z)dzW^{\varrho}(drdy)\Big|^2\Big]$$

$$+\mathbb{E}\Big[\Big|\int_0^t\int\int q^{\epsilon}_{t-r}(x,z)\Big(a(r,z,u(r,z))-a(r,y,u(r,y))\Big)p_\epsilon(y,z)dzW^{\varrho}(drdy)\Big|^2\Big]$$

$$+\mathbb{E}\Big[\Big|\int_0^t\int\Big(\int_{\mathbb{R}}q^{\epsilon}_{t-r}(x,z)p_\epsilon(y,z)dz-p_{t-r}(x,y)\Big)a(r,y,u(r,y))W^{\varrho}(drdy)\Big|^2\Big]\ \Big\}$$

$$=:\ e^{-\lambda|x|}\ 2\Big\{I^{\epsilon}_1(t,x)+\cdots+I^{\epsilon}_9(t,x)\Big\}.$$

By Lemma 4.2, Lemma 4.4, Lemma 6.14 ($i=2$, $\xi=z$), Lemma 6.15 and Hölder's inequality we obtain for $t\le T$:

$$\begin{aligned}
I^{\epsilon}_6(t,x) &= \mathbb{E}\Big[\int_0^t\int_{\mathbb{R}}e^{-2\frac{t-r}{\epsilon}}a^2(r,x,u_\epsilon(r,x))p^2_\epsilon(x,y)\varrho(drdy)\Big]\\
&\le c\int_0^t e^{-2\frac{t-r}{\epsilon}}\int_{\mathbb{R}}p^2_\epsilon(x,y)e^{\lambda|x|}\varrho_1(r,dy)\tilde c(1+\|u_\epsilon\|_{\lambda,r,1})^2\varrho_2(dr)\\
&\le c\int_0^t e^{-2\frac{t-r}{\epsilon}}\frac{1}{2\pi\epsilon}\int_{\mathbb{R}}e^{-\frac{(x-y)^2}{\epsilon}}\varrho_1(r,dy)\ \tilde c_{\lambda,T}\ \varrho_2(dr)\ e^{\lambda|x|}\\
&\le \tilde c_{\lambda,T}\ \frac{1}{\epsilon^{1-\alpha_1/2}}\int_0^t e^{-2\frac{t-r}{\epsilon}}\varrho_2(dr)\ e^{\lambda|x|}\ \le\ c_{\lambda,T}\ \epsilon^{\alpha_1/2+\alpha_2-1}\ e^{\lambda|x|},
\end{aligned}$$

$$\begin{aligned}
I^{\epsilon}_7(t,x) &= \mathbb{E}\Big[\int_0^t\int_{\mathbb{R}}\Big(\int_{\mathbb{R}}q^{\epsilon}_{t-r}(x,z)\\
&\qquad\times\Big(a(r,z,u_\epsilon(r,z))-a(r,z,u(r,z))\Big)p_\epsilon(y,z)dz\Big)^2\varrho(drdy)\Big]\\
&\le c\,\mathbb{E}\Big[\int_0^t\int_{\mathbb{R}}\ \int_{\mathbb{R}}q^{\epsilon}_{t-r}(x,z)^2e^{\lambda|z|}p_\epsilon(y,z)dz\\
&\qquad\times\int_{\mathbb{R}}p_\epsilon(y,z)e^{-\lambda|z|}|u_\epsilon(r,z)-u(r,z)|^2dz\ \ \varrho(drdy)\Big]\\
&\le c\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}}q^{\epsilon}_{t-r}(x,z)^2e^{\lambda|z|}p_\epsilon(y,z)dz\varrho_1(r,dy)\ \|u_\epsilon-u\|^2_{\lambda,r,1}\ \varrho_2(dr)\\
&\le c_{\lambda,T}\int_0^t\frac{1}{(t-r)^{1-\alpha_1/2}}\|u_\epsilon-u\|^2_{\lambda,r,1}\ \varrho_2(dr)\ e^{\lambda|x|}
\end{aligned}$$

and

$$I^{\epsilon}_8(t,x)\ =\ \mathbb{E}\Big[\int_0^t\int_{\mathbb{R}}\Big(\int_{\mathbb{R}}q^{\epsilon}_{t-r}(x,z)$$

$$
\begin{aligned}
&\qquad\qquad \times\Big(a(r,z,u(r,z)) - a(r,y,u(r,y))\Big)p_\epsilon(y,z)dz\Big)^2 \varrho(drdy)\Big]\\
\le\ & c\int_0^t \int_{\mathbb{R}}\int_{\mathbb{R}} q^\epsilon_{t-r}(x,z)^2 e^{\lambda|z|} p_\epsilon(y,z)dz\\
&\quad \times \int_{\mathbb{R}} p_\epsilon(y,z) 2L_t^2\Big(|z-y|^{2\kappa} + \mathbb{E}\Big[|u(r,z)-u(r,y)|^2\Big]\Big)e^{-\lambda|z|}dz\ \varrho(drdy)\\
\le\ & c'_T\int_0^t \int_{\mathbb{R}}\int_{\mathbb{R}} q^\epsilon_{t-r}(x,z)^2 e^{\lambda|z|} p_\epsilon(y,z)dz\\
&\quad \times \int_{\mathbb{R}} p_\epsilon(y,z)\Big(|z-y|^{2\kappa} + c_T|z-y|^{2\alpha}e^{\lambda|z-y|}e^{\lambda|z|}\Big)e^{-\lambda|z|}dz\ \varrho(drdy)\\
\le\ & c''_T\int_0^t \int_{\mathbb{R}}\int_{\mathbb{R}} q^\epsilon_{t-r}(x,z)^2 e^{\lambda|z|} p_\epsilon(y,z)dz\\
&\quad \times\Big(\int_{\mathbb{R}} p_\epsilon(y,z)|z-y|^{2\kappa}dz + \int_{\mathbb{R}} p_\epsilon(y,z)|z-y|^{2\alpha}e^{\lambda|z-y|}dz\Big)\varrho(drdy)\\
\le\ & c_{\lambda,T}\, e^{\lambda|x|}\,(\epsilon^{\kappa/2}+\epsilon^{\alpha/2}).
\end{aligned}
$$

For the estimate of I_8 we used

$$
\mathbb{E}[|u(r,z)-u(r,y)|^2] \le c_T|z-y|^{2\alpha}e^{\lambda|z-y|}e^{\lambda|z|} \quad \forall\, r\le T \text{ and } z,y\in\mathbb{R}
$$

which follows from Step 1 of the proof of Theorem 6.8. Further, using Lemma 6.16 and Lemma 6.14 ($i=1$, $\xi=y$) we can estimate $I_9^\epsilon(t,x)$ by

$$
\begin{aligned}
&\mathbb{E}\Big[\int_0^t \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} q^\epsilon_{t-r}(x,z)p_\epsilon(y,z)dz - p_{t-r}(x,y)\Big)^2 a^2(r,y,u(r,y))\varrho(drdy)\Big]\\
&\le\ c\int_0^t \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} q^\epsilon_{t-r}(x,z)p_\epsilon(y,z)dz - p_{t-r}(x,y)\Big)^2\\
&\qquad\qquad\qquad\qquad \times e^{\lambda|y|}e^{-\lambda|y|}\mathbb{E}\Big[(1+u(r,y))^2\Big]\varrho(drdy)\\
&\le\ c'\int_0^t \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} q^\epsilon_{t-r}(x,z)p_\epsilon(y,z)dz - p_{t-r}(x,y)\Big)^2 e^{\lambda|y|}\varrho_1(r,dy)\|u\|^2_{\lambda,r,1}\varrho_2(dr)\\
&\le\ c'_T\int_0^t \int_{\mathbb{R}} \Big|\int_{\mathbb{R}} q^\epsilon_{t-r}(x,z)p_\epsilon(y,z)dz - p_{t-r}(x,y)\Big|\\
&\qquad\qquad \times\Big(\int_{\mathbb{R}} q^\epsilon_{t-r}(x,z)p_\epsilon(y,z)dz + p_{t-r}(x,y)\Big)e^{\lambda|y|}\ \varrho_1(r,dy)\varrho_2(dr)\\
&\le\ c'_T\int_0^t \int_{\mathbb{R}} c_{\delta,T}\frac{1}{(t-r)^{1/2+\delta}}\epsilon^\delta \times c_T\frac{1}{(t-r)^{1/2-\alpha_1/2}}e^{\lambda|x|}\varrho_2(dr) \le\ c'_{\delta,T}\, e^{\lambda|x|}\,\epsilon^\delta
\end{aligned}
$$

for some $\delta\in(0,\alpha_1/2+\alpha_2-1)$.

Proceeding in the same way we get analogous estimates for $I_2^\epsilon(t,x)$, $I_3^\epsilon(t,x)$, $I_4^\epsilon(t,x)$ and $I_5^\epsilon(t,x)$. Also, by Lemma 6.12 we have $\lim_{\epsilon\downarrow 0} I_1^\epsilon(t,x)=0$. On the whole, we obtain

$$
\|u_\epsilon - u\|^2_{\lambda,t,1} \ \le\ c_{\lambda,T}\Big\{\ h_T(\epsilon)\ +\ \sup_{s\in[0,t]}\int_0^s \frac{1}{(s-r)^{1-\alpha_1/2}}\|u_\epsilon-u\|^2_{\lambda,r,1}\ \varrho_2(dr)
$$

$$+ \sup_{s\in[0,t]} \int_0^s \frac{1}{(s-r)^{1/2-\beta_1/2}} \|u_\epsilon - u\|^2_{\lambda,r,1}\, \sigma_2(dr)\Big\}$$

for all $t \leq T$, for any $\lambda > 0$ and some $h_T(.)$ satisfying $h_T(\epsilon) \downarrow 0$ as $\epsilon \downarrow 0$. Lemma 4.12 then gives $\|u_\epsilon - u\|_{\lambda,T,1} \leq \tilde{c}_{\lambda,T}\, h_T(\epsilon)$ ($\downarrow 0$ as $\epsilon \downarrow 0$) for every $T > 0$. Since u_ϵ and u are jointly continuous and u_ϵ is non-negative for every $\epsilon > 0$, u is non-negative, too. We are done. □

6.5 Weak solutions (continuous coefficients)

In Section 6.3 we constructed strong solutions to SPDE (5.9) by means of a Picard-Lindelöf iteration. The key was the Lipschitz continuity of the coefficients. If the coefficients fail to be Lipschitz continuous, then the arguments do not apply any more. However, for any continuous coefficients, that grow at most linearly, we can find at least weak solutions (Theorem 6.17). The key is a tightness argument. For simplicity we assume $a(t,x,u)$ and $b(t,x,u)$ to be independent of t and x. In the Lipschitz case strong uniqueness of solutions could be obtained comparatively easily. In the non-Lipschitz case, however, the question of uniqueness becomes much more delicate. While statements on strong uniqueness do not exist so far, weak uniqueness could be established for the following setting: $\varrho(dtdx) = dtdx$, $b \equiv 0$, $a(t,x,u) = u^\gamma$ and $\gamma \in [\frac{1}{2}, 1)$. For the case $\gamma = \frac{1}{2}$ see [RC86], the case $\gamma \in (\frac{1}{2}, 1)$ was studied in [Myt98]. Below (Theorem 6.21) we give a generalization of the result on $\gamma = \frac{1}{2}$. The interest in this case is due to the relation to the catalytic super-Brownian motion, cf. Section 9.8. Solutions to SPDE (5.9) were defined in Definition 5.20.

Theorem 6.17 [WEAK $C_{tem}(\mathbb{R})$-VALUED SOLUTION] *Let $a(t,x,u) = a(u)$, $b(t,x,u) = b(u)$ be continuous and assume there exists a finite constant $c > 0$ such that for all $u \in \mathbb{R}$:*

$$|a(u)| + |b(u)| \leq c(1 + |u|). \tag{6.24}$$

Let $\eta \in C_{tem}(\mathbb{R})$, $\varrho(dtdx)$ satisfy condition (A) *with α_1, α_2 and $\sigma(dtdx)$ satisfy condition* (B) *with β_1, β_2. Then SPDE (5.9) with initial condition η has a weak $C_{tem}(\mathbb{R})$-valued solution which is locally jointly Hölder-γ-continuous for all $\gamma \in (0, \frac{\alpha}{2} \wedge \beta)$, where $\alpha := \frac{\alpha_1}{2} + \alpha_2 - 1$ and $\beta := \frac{\beta_1}{2} + \beta_2 - 1/2$. If in addition $\eta \in C^+_{tem}(\mathbb{R})$, $a(0) = 0$ and $b(0) \geq 0$, then this solution is $\mathbb{P}$-almost surely non-negative.*

Proof We may and do pick two sequences (a_n) and (b_n) of Lipschitz continuous functions approximating a and b, respectively, uniformly on compacts. Also, a_n and b_n can be chosen in such a manner that they fulfill (6.24) with a common constant c for all $n \geq 1$. Further, let $W^\varrho = [W^\varrho, \Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}]$ be a continuous orthogonal martingale measure with quadratic variation measure $\langle M\rangle(dtdx) = \varrho(dtdx)$. By Theorem 6.8 there is for every $n \geq 1$ a unique strong $C_{tem}(\mathbb{R})$-valued solutions u_n to SPDE (5.9) with a, b replaced by a_n, b_n. Let $(\mathbb{P}_n)$ denote the sequences of probability measures on $(C([0,\infty), C_{tem}(\mathbb{R})), d_{tem,\infty})$ induced by

(u_n)[22] and set

$$X_t^n(x) := \int_0^t \int_{\mathbb{R}} p_{t-r}(x,y) b_n(r,y,u_n(r,y))\sigma(drdy) + \int_0^t \int_{\mathbb{R}} p_{t-r}(x,y) a_n(r,y,u_n(r,y)) W^{\varrho}(drdy)$$

for every $n \geq 1$. As in Step 1 of the proof of Theorem 6.8 (Section 6.3) one can show that

$$\sup_{n\geq 1} \mathbb{E}_n\Big[|X_t^n(x) - X_{t'}^n(x')|^{2m}\Big] \leq c_{\lambda,T}\Big(|t-t'|^{(\alpha m)\wedge(\beta 2m)} + |x-x'|^{2((\alpha m)\wedge(\beta 2m))}\Big)e^{\lambda|x|}$$

holds for all $0 \leq t, t' \leq T$, $x, x' \in \mathbb{R}$ with $|x-x'| \leq 1$, $\lambda > 0$ and $m \geq 1$. Thus, for m sufficiently large, Proposition 3.15(b) implies tightness of the sequence $(\mathbb{P}_n)$. According to Prohorov's theorem (Theorem 2.15), (u_n) is relatively compact w.r.t. weak convergence. Any weak limit point is locally Hölder-γ-continuous on $(0,\infty)\times\mathbb{R}$ for each $\gamma \in (0, \frac{\alpha}{2}\wedge\beta)$ which is also a consequence of Proposition 3.15(b) (and of (4.6)). In order to complete the proof of Theorem 6.17 we only have to show yet that any weak limit point is a weak $C_{tem}(\mathbb{R})$-valued solution to SPDE (5.9). We need the following lemma.

Lemma 6.18 *For all $\lambda > 0$, $T > 0$ and $m \geq 1$ we have*

$$\sup_{n\geq 1} \|u_n\|_{\lambda,T,m} \leq c_{\lambda,T,m} < \infty$$

where $\|.\|_{\lambda,T,m}$ is defined as in Section 6.3.

Proof For every $n \geq 1$, let $(u_{n,l})_{l\geq 1}$ denote the Picard-Lindelöf approximation of u_n (cf. the proof of Theorem 6.8, Section 6.3) starting at $P_{.}\eta(.)$. Then, as in Step 2 of the proof of Theorem 6.8, one can show that

$$\|u_{n,l+1}\|_{\lambda,T,m}^{2m} \leq c_{\lambda,T,m}\Big\{1 + \sup_{t\in[0,T]} \int_0^t \frac{1}{(t-r)^{1-\alpha_1/2}} \|u_{n,l}\|_{\lambda,r,m}^{2m} \varrho_2(dr) + \sup_{t\in[0,T]} \int_0^t \frac{1}{(t-r)^{1/2-\beta_1/2}} \|u_{n,l}\|_{\lambda,r,m}^{2m} \sigma_2(dr)\Big\}$$

holds for all $n, l \geq 1$. Moreover, the constant $c_{\lambda,T,m}$ can be chosen to be universal (for all n, l) since (6.24) was assumed to be fulfilled with a common constant c (for all n, l). The Gronwall-type Lemma 4.11 then implies $\sup_{l\geq 1}\|u_{n,l}\|_{\lambda,T,m} \leq \tilde{c}_{\lambda,T,m} < \infty$ for a common constant $\tilde{c}_{\lambda,T,m} > 0$ for all $n \geq 1$. This implies the claim. □

Let u denote the limit point of any weakly convergent subsequence $(u_k) \subset (u_n)$.

[22] According to Remark 3.7 there exists a law $\bar{\mathbb{P}}_n$ on $[C([0,\infty), C_{tem}(\mathbb{R})), \mathcal{B}_{\infty,\infty}]$ which can be identified with the law of u_n. Here $\mathcal{B}_{\infty,\infty}$ denotes the Borel σ-algebra w.r.t. the topology of uniform convergence on compacts (which is contained in the Borel σ-algebra $\mathcal{B}_{tem,\infty}$ w.r.t. $d_{tem,\infty}$ by continuity). Then set $\mathbb{P}_n(H) := \inf\{\sum_{i=1}^{\infty}\bar{\mathbb{P}}_n(\bar{H}_i) : \bar{H}_i \in \mathcal{B}_{\infty,\infty}, H \subset \cup_{i=1}^{\infty}\bar{H}_i\}$ for $H \subset C([0,\infty), C_{tem}(\mathbb{R}))$. $\mathbb{P}_n$ provides an outer measure (Caratheodory's construction). One can show that $\mathbb{P}_n(H\cup H') = \mathbb{P}_n(H) + \mathbb{P}_n(H')$ holds for all positive separated (w.r.t. $d_{tem,\infty}$) sets H, H'. This is equivalent to $\mathbb{P}_n$ being Borel w.r.t. $\mathcal{B}_{tem,\infty}$. So $\mathbb{P}_n$ is a measure on $\mathcal{B}_{tem,\infty}$. Further, $\mathbb{P}_n = \bar{\mathbb{P}}_n$ on $\mathcal{B}_{\infty,\infty}$ and $\mathbb{P}_n$'s $\mathbb{R}$-valued coordinate process is a version of u_n.

Lemma 6.19 *For all $\lambda > 0$, $T > 0$ and $m \geq 1$, we have $\|u\|_{\lambda,T,m} \leq c_{\lambda,T,m} < \infty$.*

Proof Note that the map

$$(C([0,\infty), C_{tem}(\mathbb{R})), d_{tem,\infty}) \to (\mathbb{R}, |.|), \qquad \phi \mapsto \phi(t,x)^{2m} \wedge N$$

is in $C_b(C([0,\infty), C_{tem}(\mathbb{R}))$ for every $t \geq 0$, $x \in \mathbb{R}$, $m \geq 1$ and $N \geq 1$. By the joint continuity of u, Fatou's lemma, the weak convergence of u_k to u and Lemma 6.18 we get

$$\begin{aligned}
\|u\|^2_{\lambda,T,m} &= \sup_{t\leq T} \sup_{x\in\mathbb{R}} e^{-\lambda|x|} \mathbb{E}[u(t,x)^{2m}] \\
&= \sup_{t\leq T} \sup_{x\in\mathbb{R}} e^{-\lambda|x|} \mathbb{E}[\lim_{N\uparrow\infty} u(t,x)^{2m} \wedge N] \\
&\leq \sup_{t\leq T} \sup_{x\in\mathbb{R}} e^{-\lambda|x|} \liminf_{N\uparrow\infty} \mathbb{E}[u(t,x)^{2m} \wedge N] \\
&\leq \sup_{t\leq T} \sup_{x\in\mathbb{R}} e^{-\lambda|x|} \liminf_{N\uparrow\infty} \lim_{k\to\infty} \mathbb{E}_k[u_k(t,x)^{2m} \wedge N] \\
&\leq \sup_{t\leq T} \sup_{x\in\mathbb{R}} \sup_{k\geq 1} e^{-\lambda|x|} \mathbb{E}_k[u_k(t,x)^{2m}] \\
&\leq \sup_{t\leq T} \sup_{x\in\mathbb{R}} \sup_{k\geq 1} \|u_k\|_{\lambda,T,m} \\
&= \sup_{k\geq 1} \|u_k\|_{\lambda,T,m} \;\leq\; c_{\lambda,T,m} \;<\; \infty
\end{aligned}$$

for all $\lambda > 0$, $T > 0$ and $m \geq 1$. □

For $\psi \in C_c^\infty(\mathbb{R})$, let $M_.(\psi)$ be defined as in the (a, b, η)-martingale problem posed in Definition 6.1. Note that $M_.(\psi)$ is an $(\mathcal{F}_t^u)$-martingale if and only if

$$\begin{aligned}
0 \;=\; \mathbb{E}\Big[\Big(&\langle u(t+s,.), \psi\rangle - \langle u(t,.), \psi\rangle - \int_t^{t+s} \langle u(r,.), \frac{1}{2}\Delta\psi\rangle dr \\
&- \int_t^{t+s} \int_{\mathbb{R}} b(u(r,y))\psi(y)\sigma(drdy)\Big) \prod_{i=1}^{l} h_i(u(t_i,.))\Big]
\end{aligned} \tag{6.25}$$

holds for all $0 \leq t_1 < \cdots < t_l \leq t$, $s \geq 0$, $l \geq 1$ and $h_1, \ldots, h_l \in C_b(C_{tem}(\mathbb{R}))$. We now show that $M_.(\psi)$ is an $(\mathcal{F}_t^u)$-martingale by establishing the validity of (6.25). To this end we first prove the following lemma.

Lemma 6.20 *Let $l \geq 1$, $t \geq 0$, $t_i \geq 0$, $h_i \in C_b(C_{tem}(\mathbb{R}))$ $(i = 1, \ldots, l)$, $\psi \in C_c^\infty(\mathbb{R})$ and consider the following maps from $(C([0,\infty), C_{tem}(\mathbb{R})), d_{tem,\infty})$ to $\mathbb{R}$:*

$f_1 : \phi \mapsto \langle \phi(t,.), \psi\rangle \prod_{i=1}^l h_i(\phi(t_i,.))$,

$f_2 : \phi \mapsto \int_0^t \langle \phi(r,.), \frac{1}{2}\Delta\psi\rangle dr \prod_{i=1}^l h_i(\phi(t_i,.))$,

$f_3 : \phi \mapsto \int_0^t \int_{\mathbb{R}} b(\phi(r,y))\psi(y)\sigma(drdy) \prod_{i=1}^l h_i(\phi(t_i,.))$,

$f_{3,k} : \phi \mapsto \int_0^t \int_{\mathbb{R}} b_k(\phi(r,y))\psi(y)\sigma(drdy) \prod_{i=1}^l h_i(\phi(t_i,.))$.

Then,

$$\lim_{k\to\infty} \mathbb{E}_k[f_i(u_k)] = \mathbb{E}[f_i(u)] \quad (i=1,2), \qquad \lim_{k\to\infty} \mathbb{E}_k[f_{3,k}(u_k)] = \mathbb{E}[f_3(u)].$$

Proof For every $N \geq 1$ define the function

$$F_N : \mathbb{R} \to \mathbb{R}, \quad x \mapsto -N\mathbf{1}_{(-\infty,-N)}(x) + x\mathbf{1}_{[-N,N]}(x) + N\mathbf{1}_{(N,\infty)}(x).$$

Step 1. We first show

$$\begin{aligned} \lim_{N\uparrow\infty} \mathbb{E}_k[f_i(F_N(u_k))] &= \mathbb{E}_k[f_i(u_k)] \quad \text{uniformly in } k \geq 1, \quad i=1,2 \\ \lim_{N\uparrow\infty} \mathbb{E}_k[f_{3,k}F_N((u_k))] &= \mathbb{E}_k[f_{3,k}(u_k)] \quad \text{uniformly in } k \geq 1. \end{aligned} \tag{6.26}$$

Since we assumed (6.24), we may assume the existence of a constant $\tilde{c} > 0$ such that $\sup_{k\geq 1} |b_k(u)| \leq \tilde{c}|u|$ holds for all $u \in \mathbb{R}$ with $|u| \geq 1$. So we obtain the following uniform (in $k \geq 1$) estimate:

$$\begin{aligned} &\mathbb{E}_k\Big[\Big|f_{3,k}(F_N(u_k)) - f_{3,k}(u_k)\Big|\Big] \\ &\leq \mathbb{E}_k\Big[\Big|\int_0^t \int_{\mathbb{R}} \Big(b_k(F_N(u_k(r,y))) - b_k(u_k(r,y))\Big)\psi(y)\sigma(drdy)\Big|\Big]\Big(\prod_{i=1}^l \|h_i\|_\infty\Big) \\ &\leq c_h\mathbb{E}_k\Big[\int_0^t \int_{\mathbb{R}} |b_k(u_k(r,y))|\mathbf{1}_{|u_k(r,y)|>N}\psi(y)\sigma(drdy)\Big] \\ &\leq c_h\mathbb{E}_k\Big[\int_0^t \int_{\mathbb{R}} \tilde{c}\,|u_k(r,y)|\mathbf{1}_{|u_k(r,y)|>N}\psi(y)\sigma(drdy)\Big] \\ &\leq \tilde{c}_h \int_0^t \int_{\mathbb{R}} \psi(y)\sigma(drdy) \sup_{j\geq 1}\sup_{t\leq T}\sup_{y\in\text{supp}(\psi)} \mathbb{E}_j[|u_j(r,y)|\mathbf{1}_{|u_j(r,y)|>N}]. \end{aligned}$$

But by Lemma 3.5 and Lemma 6.18 the family $(u_j(r,y) : j \geq 1, r \leq T, y \in \text{supp}(\psi))$ is uniformly integrable, i.e. the latter bound converges to 0 as $N \uparrow \infty$. This implies the second line in (6.26). The first line, i.e. the cases $i = 1,2$, can be proved analogously.

Step 2. By the joint continuity of u we have $\lim_{N\uparrow\infty} f_i(F_N(u)) = f_i(u)$ $\mathbb{P}$-almost surely, for $i = 1,2,3$. Also, $|F_N(x)| \leq |x|$ holds for all $x \in \mathbb{R}$ and every $N \geq 1$. Therefore, we obtain by Lemma 6.19 and the dominated convergence theorem (for $i = 1,2,3$):

$$\begin{aligned} &\lim_{N\uparrow\infty} \Big|\mathbb{E}[f_i(F_N(u))] - \mathbb{E}[f_i(u)]\Big| \\ &\leq \lim_{N\uparrow\infty} \mathbb{E}\Big[|f_i(F_N(u)) - f_i(u)|\Big] = \mathbb{E}\Big[\lim_{N\uparrow\infty} |f_i(F_N(u)) - f_i(u)|\Big] = 0. \end{aligned} \tag{6.27}$$

Step 3. By assumption, $\lim_{k\to\infty} \|b_k(F_N(.)) - b(F_N(.))\|_\infty = 0$ holds and so we obtain

$$\lim_{k\to\infty} \Big|\mathbb{E}_k[f_{3,k}(F_N(u_k))] - \mathbb{E}_k[f_3(F_N(u_k))]\Big| = 0. \tag{6.28}$$

Further, the maps $f_1(F_N)$, $f_2(F_N)$ and $f_3(F_N)$ are clearly in $C_b(C([0,\infty), C_{tem}(\mathbb{R})))$ for every $N \geq 1$. So, since u_k converges weakly to u, we obtain for every $N \geq 1$:

$$\lim_{k\to\infty} \mathbb{E}_k[f_i(F_N(u_k))] = \mathbb{E}[f_i(F_N(u))], \qquad i = 1, 2, 3. \tag{6.29}$$

In particular, (6.28) and the convergence in (6.29) for $i = 3$ yield

$$\lim_{k\to\infty} \mathbb{E}_k[f_{3,k}(F_N(u_k))] = \mathbb{E}[f_3(F_N(u))]. \tag{6.30}$$

Step 4. By (6.26), (6.27) and (6.29) we get

$$\begin{aligned}\lim_{k\to\infty} \mathbb{E}_k[f_i(u_k)] &= \lim_{k\to\infty} \lim_{N\uparrow\infty} \mathbb{E}_k[f_i(F_N(u_k))] \\ &= \lim_{N\uparrow\infty} \lim_{k\to\infty} \mathbb{E}_k[f_i(F_N(u_k))] = \lim_{N\uparrow\infty} \mathbb{E}[f_i(F_N(u))] = \mathbb{E}[f_i(u)], \qquad i = 1, 2.\end{aligned}$$

Moreover, (6.26), (6.27) and (6.30) yield

$$\begin{aligned}\lim_{k\to\infty} \mathbb{E}_k[f_{3,k}(u_k)] &= \lim_{k\to\infty} \lim_{N\uparrow\infty} \mathbb{E}_k[f_{3,k}(F_N(u_k))] \\ &= \lim_{N\uparrow\infty} \lim_{k\to\infty} \mathbb{E}_k[f_{3,k}(F_N(u_k))] = \lim_{N\uparrow\infty} \mathbb{E}[f_3(F_N(u))] = \mathbb{E}[f_3(u)].\end{aligned}$$

This proves the claim of Lemma 6.20. □

By Proposition 6.4, u_k solves the (a_k, b_k, η)-martingale problem. So (6.25) with $\mathbb{E}, u, a, b$ replaced by $\mathbb{E}_k, u_k, a_k, b_k$ holds. Also, according to Lemma 6.20, the r.h.s. of (6.25) with $\mathbb{E}, u, a, b$ replaced by $\mathbb{E}_k, u_k, a_k, b_k$ converges to the r.h.s. of (6.25) as $k \to \infty$. Consequently, (6.25) holds and so

$$M_t(\psi) := \langle u(t,.), \psi\rangle - \langle \eta, \psi\rangle - \int_0^t \langle u(r,.), \frac{1}{2}\Delta\psi\rangle dr - \int_0^t \int_{\mathbb{R}} b(u(r,y))\psi(y)\sigma(drdy)$$

is an $(\mathcal{F}_t^u)$-martingale. Using Lemma 6.19 and the joint continuity of u, it is easy to verify that $M_.(\psi)$ is continuous and square-integrable. By Proposition 3.23, it is also a continuous square-integrable martingale w.r.t. the usual augmentation $(\bar{\mathcal{F}}_t^u)$ of $(\mathcal{F}_t^u)$; recall that $(\bar{\mathcal{F}}_t^u)$ satisfies the usual conditions.

By the uniqueness of the Doob-Meyer decomposition (cf. Theorem 3.22), $M_t(\psi)$ has quadratic variation process

$$\langle M(\psi)\rangle_t = \int_0^t \int_{\mathbb{R}} a^2(u(r,y))\psi^2(y)\varrho(drdy)$$

if and only if

$$0 = \mathbb{E}\Big[\Big(M_{t+s}^2(\psi) - M_t^2(\psi) - \int_t^{t+s}\int_{\mathbb{R}} a^2(u(r,y))\psi^2(y)\varrho(drdy)\Big)\prod_{i=1}^l h_i(u(t_i,.))\Big] \tag{6.31}$$

for all $0 \leq t_1 < \cdots < t_l \leq t$, $s \geq 0$, $l \geq 1$ and $h_1, \ldots, h_l \in C_b(C_{tem}^+(\mathbb{R}))$. Now, the analogue of (6.31) for u_k, a_k, b_k holds. Using techniques as for the verification of (6.25) we can show that the r.h.s. of this analogue converges to the r.h.s. of (6.31) as $k \to \infty$. Hence, (6.31) holds and so u is a solution to the (a, b, η)-martingale problem. Proposition 6.4 then shows that u is a weak solution to SPDE (5.9).

If $\eta \in C^+_{tem}(\mathbb{R})$, $a(0) = 0$ and $b(0) \geq 0$, then each u_n of the sequence (u_n) is non-negative. Note that the tightness criteria in Propositions 3.13-3.15 remain true for probability measures on $C([0,\infty), C^+_{tem}(\mathbb{R}^d))$. So the above procedure provides some probability measure on $C([0,\infty), C^+_{tem}(\mathbb{R}^d))$ whose coordinate process u solves SPDE (5.9) weakly. This completes the proof of Theorem 6.17. □

Now we turn to the uniqueness claim for the coefficients $a(t,x,u) = \sqrt{|u|}$ and $b \equiv 0$. According to Theorem 6.17, the corresponding SPDE has a weak $C^+_{tem}(\mathbb{R})$-valued solution whenever the initial condition $\eta \in C_{tem}(\mathbb{R})$ is non-negative. The next theorem shows that this solution is also weakly unique among $C^+_{tem}(\mathbb{R})$-valued solutions.

Theorem 6.21 [WEAK UNIQUENESS] *Let $a(t,x,u) = \sqrt{|u|}$, $b \equiv 0$, $\eta \in C^+_{tem}(\mathbb{R})$ and $\varrho(dtdx)$ satisfy condition* (A). *Then the weak $C^+_{tem}(\mathbb{R})$-valued solution to SPDE (5.9) with initial condition η is weakly unique among $C^+_{tem}(\mathbb{R})$-valued solutions.*

We will not prove Theorem 6.21 at this place since a very similar result will thoroughly be proved in Section 9.9 (resp. 9.7). There the state space of $u = (u(t,.) : t \geq 0)$ is $C^+_{int}(\mathbb{R})$ (resp. $\mathcal{M}_f(\mathbb{R})$) instead of $C^+_{tem}(\mathbb{R})$. However, the adaption of the proof to the present setting is not difficult when taking note of Lemma 6.2 of [DP98]. This lemma (with $Y \equiv 0$) says: Two laws $\mathbb{P}, \mathbb{P}'$ on $[C^+_{tem}(\mathbb{R}), \mathcal{B}^+_{tem}]$ coincide if $\int e^{-\langle \phi,\psi \rangle}\mathbb{P}(d\phi) = \int e^{-\langle \phi,\psi \rangle}\mathbb{P}'(d\phi)$ $\forall \psi \in C_c(\mathbb{R})$, where $\mathcal{B}^+_{tem}$ denotes the Borel σ-algebra w.r.t. d_{tem}. In particular, the state space of u's dual process should be chosen to be $C^+_{rap}(\mathbb{R})$ instead of $C^+_b(\mathbb{R})$.

6.6 Refinement of the state space

So far the state space of solutions to SPDE (5.9) was always be assumed to be $C_{tem}(\mathbb{R})$. Occasionally one wishes to work with the finer state space $C_{rap}(\mathbb{R})$. In this section we show that a solution stays in $C^+_{rap}(\mathbb{R})$ whenever the initial state η is an element of $C^+_{rap}(\mathbb{R})$. We require the solutions u to satisfy:

$$\sup_{t \leq T} \sup_{x \in \mathbb{R}} e^{-\lambda|x|}\mathbb{E}\Big[|u(t,x)|^{2m}\Big] < \infty \qquad \forall \lambda, T > 0 \text{ and } m \geq 1. \tag{6.32}$$

However, this is no real restriction since any solution u, that is given by Theorem 6.8 or Theorem 6.17, satisfies (6.32); cf. the corresponding proofs.

Theorem 6.22 [SOLUTION PRESERVING THE STATE SPACE $C^+_{rap}(\mathbb{R})$] *Assume for every $T > 0$ there are constants $\theta \in (0,1)$ and $c_T > 0$ such that for every $t \leq T$, $x \in \mathbb{R}$ and $u \in \mathbb{R}$:*

$$|a(t,x,u)| \leq c_T(|u| + |u|^\theta), \qquad |b(t,x,u)| \leq c_T|u|. \tag{6.33}$$

Let $\varrho(dtdx)$ and $\sigma(dtdx)$ satisfy condition (A) *and* (B)*, respectively. Then every (strong or weak) $C^+_{tem}(\mathbb{R})$-valued solution to SPDE (5.9) with initial condition $\eta \in C^+_{rap}(\mathbb{R})$, that satisfies (6.32), is even $C^+_{rap}(\mathbb{R})$-valued continuous.*

Proof Let u be a $C^+_{tem}(\mathbb{R})$-valued solution to SPDE (5.9) with initial condition $\eta \in C^+_{rap}(\mathbb{R})$. It does not matter whether u is a strong or a weak solution.

Step 1. We first show (for every $\lambda, T > 0$) that

$$\sup_{t\leq T}\sup_{x\in\mathbb{R}} e^{\lambda|x|}\mathbb{E}\Big[|u(t,x)|\Big] < \infty \tag{6.34}$$

holds whenever $\eta \in C^+_{rap}(\mathbb{R})$. Recall that Walsh-integrals are mean zero martingales in time. Also, u is non-negative by assumption, i.e. $|u(\omega)| = u(\omega)$ for $\mathbb{P}$-almost all ω. So we obtain by means of (6.33):

$$\begin{aligned}
\mathbb{E}\Big[|u(t,x)|\Big] &= \Big|\mathbb{E}[u(t,x)]\Big| \\
&= \Big|P_t\eta(x) + \int_0^t\int_{\mathbb{R}} p_{t-r}(x,y)\mathbb{E}\Big[b(r,y,u(r,y))\Big]\sigma(drdy)\Big| \\
&\leq P_t\eta(x) + \int_0^t\int_{\mathbb{R}} p_{t-r}(x,y)\mathbb{E}\Big[\,|b(r,y,u(r,y))|\,\Big]\sigma(drdy) \\
&\leq P_t\eta(x) + \sup_{s\leq t}\int_0^s\int_{\mathbb{R}} p_{s-r}(x,y)\mathbb{E}\Big[|u(r,y)|\Big]\sigma(drdy)
\end{aligned}$$

for all $t \geq 0$ and $x \in \mathbb{R}$. Since $P.\eta$ is $C^+_{rap}(\mathbb{R})$-valued continuous (recall Lemma 4.9), the Gronwall-type Lemma 4.13 and (6.32) (with $m = 1$) imply (6.34).

Step 2. We next show

$$\sup_{t\leq T}\sup_{x\in\mathbb{R}} e^{\lambda|x|}\mathbb{E}\Big[|u(t,x)|^{2m}\Big] < \infty \tag{6.35}$$

for every $\lambda, T > 0$ and $m \geq 1$. To do so it is enough to show

$$\sup_{t\leq T}\sup_{x\in\mathbb{R}} e^{\lambda|x|}\mathbb{E}\Big[|u(t,x)|^{1/\theta^n}\Big] < \infty \tag{6.36}$$

for every $\lambda, T > 0$ and $n = 0, 1, 2, \ldots$. W.l.o.g. we may assume $\theta \in (0, \frac{1}{2})$. We proceed by induction on n in order to show (6.36). For $n = 0$, (6.36) equals (6.34). Now, suppose (6.36) holds for all $\lambda, T > 0$ up to some $n \geq 0$. Set $q = q(n) := \frac{1}{2\theta^{n+1}}$. By Theorem 3.28 and Hölder's inequality ($\frac{q-1}{q} + \frac{1}{q} = 1$)[23] as well as Lemmas 4.2$(i) \Rightarrow (ii)$ and 4.4(i) we get:

$$\begin{aligned}
&\mathbb{E}\Big[\Big|\int_0^t\int_{\mathbb{R}} p_{t-r}(x,y)a(r,y,u(r,y))W^{\varrho}(drdy)\Big|^{2q}\Big] \qquad (6.37)\\
&\leq c_q\mathbb{E}\Big[\Big(\int_0^t\int_{\mathbb{R}} p^2_{t-r}(x,y)a^2(r,y,u(r,y))\varrho(drdy)\Big)^q\Big] \\
&\leq c_q\mathbb{E}\Big[\Big|\Big(\int_0^t\int_{\mathbb{R}} p^2_{t-r}(x,y)\varrho(drdy)\Big)^{\frac{q-1}{q}}\Big(\int_0^t\int_{\mathbb{R}} p^2_{t-r}(x,y)|a(r,y,u(r,y))|^{2q}\varrho(drdy)\Big)^{\frac{1}{q}}\Big|^q\Big] \\
&\leq c_q\mathbb{E}\Big[\Big(c_T\, t^{\alpha_1/2-\alpha_2-1}\Big)^{q-1}\int_0^t\int_{\mathbb{R}} p^2_{t-r}(x,y)|a(r,y,u(r,y))|^{2q}\varrho(drdy)\Big] \\
&\leq c_{q,T}\int_0^t\int_{\mathbb{R}} p^2_{t-r}(x,y)\mathbb{E}\Big[|a(r,y,u(r,y))|^{2q}\Big]\varrho(drdy)
\end{aligned}$$

[23] Note that $q > 1$ since we assumed $\theta \in (0, \frac{1}{2})$.

for all $t \leq T$ and $x \in \mathbb{R}$. Using Hölder's inequality ($\frac{2q-1}{2q} + \frac{1}{2q} = 1$) and again Lemmas 4.2($i$) $\Rightarrow$ (ii) and 4.4(i), we obtain analogously:

$$\begin{aligned} & \mathbb{E}\Big[\Big|\int_0^t \int_{\mathbb{R}} p_{t-r}(x,y) b(r,y,u(r,y))\sigma(drdy)\Big|^{2q}\Big] \\ \leq\ & c_{q,T} \int_0^t \int_{\mathbb{R}} p_{t-r}(x,y) \mathbb{E}\Big[|b(r,y,u(r,y))|^{2q}\Big]\sigma(drdy). \end{aligned} \tag{6.38}$$

By (6.37), (6.38) and $2q = 1/\theta^{n+1}$, we get for all $t \leq T$ and $x \in \mathbb{R}$:

$$\begin{aligned} & \mathbb{E}\Big[|u(t,x)|^{1/\theta^{n+1}}\Big] \\ \leq\ & c_n \Big(P_t \eta(x)\Big)^{1/\theta^{n+1}} \\ & + c_n \mathbb{E}\Big[\Big|\int_0^t \int_{\mathbb{R}} p_{t-r}(x,y) b(r,y,u(r,y))\sigma(drdy)\Big|^{1/\theta^{n+1}}\Big] \\ & + c_n \mathbb{E}\Big[\Big|\int_0^t \int_{\mathbb{R}} p_{t-r}(x,y) a(r,y,u(r,y)) W^{\varrho}(drdy)\Big|^{1/\theta^{n+1}}\Big] \\ \leq\ & c_n \Big(P_t \eta(x)\Big)^{1/\theta^{n+1}} \\ & + c_{n,T} \int_0^t \int_{\mathbb{R}} p_{t-r}(x,y) \mathbb{E}\Big[|b(r,y,u(r,y))|^{1/\theta^{n+1}}\Big]\sigma(drdy) \\ & + c_{n,T} \int_0^t \int_{\mathbb{R}} p^2_{t-r}(x,y) \mathbb{E}\Big[|a(r,y,u(r,y))|^{1/\theta^{n+1}}\Big]\varrho(drdy). \end{aligned}$$

By (6.33) we may continue

$$\begin{aligned} \leq\ & c_n \Big(P_t \eta(x)\Big)^{1/\theta^{n+1}} \\ & + c_{n,T} \int_0^t \int_{\mathbb{R}} p_{t-r}(x,y) \mathbb{E}\Big[c_T |u(r,y)|^{1/\theta^{n+1}}\Big]\sigma(drdy) \\ & + c_{n,T} \int_0^t \int_{\mathbb{R}} p^2_{t-r}(x,y) \mathbb{E}\Big[c_T \Big(|u(r,y)| + |u(r,y)|^{\theta}\Big)^{1/\theta^{n+1}}\Big]\varrho(drdy) \\ \leq\ & c_n \Big(P_t \eta(x)\Big)^{1/\theta^{n+1}} + c'_{n,T} \int_0^t \int_{\mathbb{R}} p^2_{t-r}(x,y) \mathbb{E}\Big[|u(r,y)|^{1/\theta^{n}}\Big]\varrho(drdy) \\ & + c'_{n,T} \int_0^t \int_{\mathbb{R}} p_{t-r}(x,y) \mathbb{E}\Big[|u(r,y)|^{1/\theta^{n+1}}\Big]\sigma(drdy) \\ & + c'_{n,T} \int_0^t \int_{\mathbb{R}} p^2_{t-r}(x,y) \mathbb{E}\Big[|u(r,y)|^{1/\theta^{n+1}}\Big]\varrho(drdy) \\ \leq\ & c_n \Big(P_t \eta(x)\Big)^{1/\theta^{n+1}} + e^{-\lambda|x|} c''_{n,T} \sup_{t \leq T} \sup_{y \in \mathbb{R}} e^{\lambda|y|} \mathbb{E}\Big[|u(t,y)|^{1/\theta^{n}}\Big] \\ & + c'_{n,T} \int_0^t \int_{\mathbb{R}} p_{t-r}(x,y) \mathbb{E}\Big[|u(r,y)|^{1/\theta^{n+1}}\Big]\sigma(drdy) \\ & + c'_{n,T} \int_0^t \int_{\mathbb{R}} p^2_{t-r}(x,y) \mathbb{E}\Big[|u(r,y)|^{1/\theta^{n+1}}\Big]\varrho(drdy). \end{aligned}$$

For the last "$\le$" we used

$$\int_0^t \int_{\mathbb{R}} p_{t-r}^2(x,y) e^{-\lambda|y|} \varrho(dr dy) = c_T''\, e^{-\lambda|x|}$$

which holds by Lemmas 4.2$(i) \Rightarrow (iv)$ and 4.4(i). The Gronwall-type Lemma 4.13, the induction assumption, (6.32) and Lemma 4.9 then complete the step $n \to n+1$. This proves (6.36) for $\lambda, T > 0$ and $n \ge 0$. Hence, (6.35) holds for all $\lambda, T > 0$ and $m \ge 1$.

Step 3. Recall the definition of Φ_2 and Φ_3 from Section 6.3. With help of (6.35) we can show (as in Step 1 of the proof of Theorem 6.8) that $\Phi_2(u)$ and $\Phi_3(u)$ satisfy the assumptions of Proposition 3.9, and so $\Phi_2(u)$ and $\Phi_3(u)$ are $C_{rap}^+(\mathbb{R})$-valued continuous. By Lemma 4.9, $(P_t\eta(.) : t \ge 0)$ is $C_{rap}^+(\mathbb{R})$-valued continuous, too. Hence,

$$u = P_{\cdot}\eta(.) + \Phi_2(u) + \Phi_3(u)$$

is $C_{rap}^+(\mathbb{R})$-valued continuous which was the claim of Theorem 6.22. $\square$

7 Heat equation with singular drift ($d \geq 1$)

This chapter is devoted to solutions of PDE (5.16) in the sense of Definition 5.21 where we consider arbitrary $d \geq 1$. In Section 7.1 we establish the equivalence of PDE (5.16) in the sense of Definition 5.21 to a certain integral equation. This equivalence is essential for proving existence and uniqueness of solutions. In Section 7.2 we show the existence of unique $C_{tem}(\mathbb{R})$-valued solutions for Lipschitz continuous coefficients, and in Section 7.3 we shall see that under some additional assumptions these solutions are non-negative. For (non-Lipschitz) continuous coefficients we also obtain $C_{tem}(\mathbb{R})$-valued solutions (Sections 7.4) but uniqueness might fail in general. Section 7.5 is the counterpart to Section 6.6, i.e. it concerns the refinement of the state space $C_{tem}(\mathbb{R})$. The proofs of the results in Sections 7.1–7.5 rely on arguments that have already been used throughout Sections 6.2–6.6 (where we established the corresponding results on SPDE (5.9)). Essentially the proofs get even easier since we have no stochastic term anymore. Therefore we shall omit many details. In Section 7.6 we focus on unique non-negative solutions to PDE (5.16) with drift b acting only to the inside, and in Section 7.7 we reverse the time (a time reversal induces a certain semigroup property of the solutions). Recall that $b : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ was assumed to be continuous.

7.1 Corresponding integral equation

In this section we establish that any solution to PDE (5.16) is a solution to the integral equation (7.1) and vice versa. We assume that $\sigma(dtdx)$ satisfies condition (B).

Definition 7.1 [INTEGRAL EQUATION] *A deterministic $C_{tem}(\mathbb{R}^d)$-valued continuous process $u = (u(t, .) : t \geq 0)$ satisfying*

$$u(t,x) = P_t\eta(x) + \int_0^t \int_{\mathbb{R}^d} p_{t-r}(x,y) b(r,y,u(r,y))\sigma(drdy) \quad \forall t \geq 0, x \in \mathbb{R}^d \tag{7.1}$$

is called $C_{tem}(\mathbb{R}^d)$-valued solution to IE (7.1) with initial condition $\eta \in C_{tem}(\mathbb{R}^d)$. The solution is said to be unique if any two solutions u and u' coincide pointwise.

Proposition 7.2 [EQUIVALENCE OF PDE AND IE] *Assume $\sigma(dtdx)$ satisfy condition* (B). *Moreover, assume for every $T > 0$ there exists a finite constant $c_T > 0$ such that*

$$|b(t,x,u)| \leq c_T(1 + |u|)$$

holds for every $t \leq T$, $x \in \mathbb{R}^d$ and $u \in \mathbb{R}$. Then every $C_{tem}(\mathbb{R}^d)$-valued solution to PDE (5.16) with initial condition $\eta \in C_{tem}(\mathbb{R}^d)$ in the sense of Definition 5.21 is also a $C_{tem}(\mathbb{R}^d)$-valued solution to IE (7.1) with initial condition η, and vice versa.

Proof Let $C^2_{rap}(\mathbb{R}^d)$ be the space of functions $\psi \in C_{rap}(\mathbb{R}^d)$ with $\frac{\partial}{\partial x_i}\psi$, $\frac{\partial^2}{\partial x_i x_j}\psi \in C_{rap}(\mathbb{R}^d)$ $(i, j = 1, \dots, d)$. Note that $C_c^\infty(\mathbb{R}^d)$ is dense in $C^2_{rap}(\mathbb{R}^d)$ w.r.t. the metric

$$d_{rap,2}(\phi,\psi) := d_{rap}(\phi,\psi) + \sum_{i=1}^d d_{rap}(\frac{\partial}{\partial x_i}\phi, \frac{\partial}{\partial x_i}\psi) + \sum_{i,j=1}^d d_{rap}(\frac{\partial^2}{\partial x_i \partial x_j}\phi, \frac{\partial^2}{\partial x_i \partial x_j}\psi).$$

If we define the space

$$C^{1,2}_{rap}([0,\infty)\times\mathbb{R}^d) \;:=\; \Big\{f\in C^{1,2}([0,\infty)\times\mathbb{R}^d)\;: \\ (f(t,.):t\geq 0)\; C^2_{rap}(\mathbb{R}^d)\text{-valued continuous}, \\ (\frac{\partial}{\partial t}f(t,.):t\geq 0)\; C_{rap}(\mathbb{R}^d)\text{-valued continuous}\Big\},$$

then the proof goes along the lines of the proof of Proposition 6.7 (and Lemma 6.6) with $a\equiv 0$ and $C^{1,2}_{rap}([0,\infty)\times\mathbb{R}^d)$ instead of $C^{1,2}_{rap}([0,\infty)\times\mathbb{R})$. □

7.2 Solutions (Lipschitz continuous coefficient)

Here we establish the existence of a unique solution to PDE (5.16) with coefficient b that is Lipschitz continuous and grows at most linearly. Conditions (B) was introduced in Definitions 2.22, and solutions to PDE (5.16) were defined in Definition 5.21.

Theorem 7.3 [UNIQUE $C_{tem}(\mathbb{R}^d)$-VALUED SOLUTION] *Let b be continuous. Assume for every $T>0$ there exist finite constants $c_T, L_T>0$ such that*

$$|b(t,x,u)|\leq c_T(1+|u|),$$

$$|b(t,x,u)-b(t,x,u')|\leq L_T|u-u'|$$

hold for all $t\leq T$, $x\in\mathbb{R}^d$ and $u,u'\in\mathbb{R}$. Let $\eta\in C_{tem}(\mathbb{R}^d)$ and $\sigma(dtdx)$ satisfy condition (B) *with β_1,β_2. Then PDE (5.16) with initial condition η has a unique $C_{tem}(\mathbb{R}^d)$-valued solution $u(.,.)$. Moreover, if we set $\beta:=\frac{\beta_1}{2}+\beta_2-d/2$, this solution satisfies for every $0<t_0\leq t,t'\leq T$, $x,x'\in\mathbb{R}^d$ and $\lambda>0$:*

$$|u(t,x)-u(t',x')|\leq c_{\lambda,t_0,T}\Big(|t-t'|^{\beta}+|x-x'|^{2\beta}\Big)e^{\lambda|x-x'|}e^{\lambda|x|}. \tag{7.2}$$

Proof The proof is very similar to that of Theorem 6.8 (with $a\equiv 0$); in fact it gets even simpler. So we only sketch the proof. It remains to show that IE (7.1) has a unique solution. Proposition 7.2 then ensures that the same is true for PDE (5.16). As before, we equip $C([0,\infty),C_{tem}(\mathbb{R}^d))$ with the metric

$$d_{tem,\infty}(f,f'):=\sum_{k=1}^{\infty}\Big(1\,\wedge\,\sup_{t\leq k}\,d_{tem}(f(t,.),f'(t,.))\Big).$$

Recall from Section 3.5 that $C([0,\infty),C_{tem}(\mathbb{R}^d))$ is complete w.r.t. $d_{tem,\infty}$. To some extent, $[C([0,\infty),C_{tem}(\mathbb{R}^d)),d_{tem,\infty}]$ is the deterministic counterpart to the complete metric space $[\mathcal{P},d_{\mathcal{P}}]$ introduced in Section 6.3. The key of the proof is again a Picard-Lindelöf iteration. Define the functional

$$\Phi'(u)(t,x):=P_t\eta(x)+\Phi'_2(u)(t,x):=P_t\eta(x)+\int_0^t\int_{\mathbb{R}^d}p_{t-r}(x,y)b(r,y,u(r,y))\sigma(drdy)$$

as well as $u_0 := P_. \eta(.)$ and $u_{n+1} := \Phi'(u_n)$ for $n \geq 0$. As in Steps 1 and 2 of the proof of Theorem 6.8 (with $a \equiv 0$) one can show that for every $u \in C([0,\infty), C_{tem}(\mathbb{R}^d))$:

$$|\Phi_2'(u)(t,x) - \Phi_2'(u)(t',x')| \leq c_{\lambda,T} \left(|t-t'|^{\beta} + |x-x'|^{2\beta}\right) e^{\lambda|x-x'|} e^{\lambda|x|} \tag{7.3}$$

and $|\Phi_2'(u)(t,x)| \leq c_{\lambda,T} e^{\lambda|x|}$ hold for every $0 \leq t, t' \leq T$, $x, x' \in \mathbb{R}^d$ and $\lambda > 0$. With help of Lemma 4.6 we conclude that Φ' maps $C([0,\infty), C_{tem}(\mathbb{R}^d))$ into itself. Going ahead as in proof of Theorem 6.8 (with $a \equiv 0$) one can further show that (u_n) is a Cauchy sequence in the complete metric space $[C([0,\infty), C_{tem}(\mathbb{R}^d)), d_{tem,\infty}]$ (Step 3), that (u_n)'s limit $u_\infty \in C([0,\infty), C_{tem}(\mathbb{R}^d))$ solves the integral equation $u = \Phi'(u)$ (Step 4) and that the solution to $u = \Phi'(u)$ is unique (Step 5). The estimate (7.2) is a consequence of (7.3) and Lemma 4.6. □

7.3 Non-negativity of solutions

In the previous section we have seen that PDE (5.16) possesses a unique solution whenever the coefficient b is Lipschitz continuous and grows at most linearly. Under slightly stronger assumptions we obtain non-negativity of the solution.

Theorem 7.4 [NON-NEGATIVITY] *Let b be continuous and $\kappa > 0$. Assume for every $T > 0$ there are finite constants $c_T, L_T > 0$ such that for all $t \leq T$, $x, x' \in \mathbb{R}^d$ and $u, u' \in \mathbb{R}$:*

$$|b(t,x,u)| \leq c_T(1+|u|),$$

$$|b(t,x,u) - b(t,x',u')| \leq L_T(|x-x'|^{\kappa} + |u-u'|).$$

Further, let $\sigma(dtdx)$ satisfy condition (B)*. If moreover $b(t,x,0) \geq 0$ ($\forall t \geq 0, x \in \mathbb{R}^d$) and $\eta \in C_{tem}^+(\mathbb{R}^d)$, then the unique solution from Theorem 7.3 is non-negative.*

Proof The proof of Theorem 7.4 will be a slight modification of the proof of Theorem 6.9 (with $a \equiv 0$). So we only sketch the proof. For every $t \geq 0$, $x \in \mathbb{R}^d$ and $\epsilon > 0$ define:

$$\begin{aligned}
\Delta_\epsilon &:= \epsilon^{-1}(P_\epsilon - I) \\
P_t^\epsilon &:= e^{t\Delta_\epsilon} = e^{-t/\epsilon} e^{t/\epsilon P_\epsilon} = e^{-t/\epsilon} \sum_{n=0}^{\infty} \frac{(t/\epsilon)^n}{n!} P_{n\epsilon} = e^{-t/\epsilon} I + Q_t^\epsilon \\
Q_t^\epsilon f &:= \int_{\mathbb{R}^d} q_t^\epsilon(.,y) f(y) dy, \qquad q_t^\epsilon(x,y) \ := \ e^{-t/\epsilon} \sum_{n=1}^{\infty} \frac{(t/\epsilon)^n}{n!} p_{n\epsilon}(x,y).
\end{aligned}$$

As in Lemmas 6.10 and 6.11 one can show that $(P_t^\epsilon) \equiv (P_t^\epsilon)_{t\geq 0}$ provides a strongly continuous contraction semigroup of linear operators on $(C_0(\mathbb{R}^d), \|.\|_\infty)$ and that Δ_ϵ (defined on $C_0(\mathbb{R}^d)$) is the generator of (P_t^ϵ), for every $\epsilon > 0$. Also, one easily extends Lemmas 6.12, 6.14, 6.15 and 6.16 to any dimension $d \geq 1$. That means we have for all $f \in C_0(\mathbb{R}^d)$,

$t \geq 0$, $x, y \in \mathbb{R}^d$, $R > 0$, $\xi \in \{y, z\}$, $\lambda \geq 0$, $\gamma > 0$, $\delta > 0$, $T > 0$ and $\epsilon \in (0, 1]$:

$$\lim_{\epsilon \downarrow 0} \|P_t^\epsilon f - P_t f\|_\infty = 0 \tag{7.4}$$

$$\sup_{r \in [0,R]} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} q_t^\epsilon(x,z) p_\epsilon(z,y) e^{\lambda|\xi|} dz \sigma_1(r, dy) \leq \frac{c_{\lambda,R}\, e^{\lambda|x|}}{t^{d/2-\beta_1/2}} \tag{7.5}$$

$$\int_{\mathbb{R}^d} p_\epsilon(x,y)|x-y|^\gamma e^{\lambda|x-y|} dy \leq c_\lambda\, \epsilon^{\gamma/4} \tag{7.6}$$

$$\lim_{\epsilon \downarrow 0} \sup_{x,y \in \mathbb{R}^d} \sup_{t \leq T} t^{d/2+\delta} \left| \int_{\mathbb{R}^d} q_t^\epsilon(x,z) p_\epsilon(y,z) dz - p_t(x,y) \right| = 0. \tag{7.7}$$

Consider the measure $\sigma_x^\epsilon(dt) := \int_{\mathbb{R}^d} p_\epsilon(x,y)\sigma(dtdy)$ and note that $\sigma_x^\epsilon(t) := \int_0^t \sigma_x^\epsilon(dr)$ is a non-decreasing continuous function, for every $\epsilon > 0$. Then we can follow the lines of the proof of Theorem 6.9 (with $a \equiv 0$) where one has to use (7.4), (7.5), (7.6) and (7.7) instead of Lemmas 6.12, 6.14, 6.15 and 6.16. First one shows that, for every fixed $\epsilon > 0$, the following equation family with index $x \in \mathbb{R}^d$

$$u_\epsilon(t,x) = \eta(x) + \int_0^t \frac{1}{2}\Delta_\epsilon u_\epsilon(r,x) dr + \int_0^t b(r,x,u_\epsilon(r,x))\sigma_x^\epsilon(dr)$$

has a unique $C_{tem}(\mathbb{R}^d)$-valued continuous solution u_ϵ (Step 1). Then one establishes that u^ϵ is non-negative (Step 2). Finally one approximates the unique solution u of PDE (5.16) by u_ϵ as $\epsilon \downarrow 0$ (Step 3) whereby the desired non-negativity of u follows. □

7.4 Solutions (continuous coefficient)

In Section 7.2 we constructed solutions to PDE (5.16) by means of a Picard-Lindelöf iteration. The key was the Lipschitz continuity of the coefficient b. If the coefficient fails to be Lipschitz continuous, then the arguments do not apply any more. Nevertheless we can find a solution for any continuous coefficient b that grows at most linearly (Theorem 7.5). The key is a compactness argument. Note, however, that uniqueness of solutions might fail in general. For simplicity we assume $b(t,x,u)$ to be independent of t and x, i.e. $b : \mathbb{R} \to \mathbb{R}$. Solutions to PDE (5.16) were defined in Definition 5.21.

Theorem 7.5 [$C_{tem}(\mathbb{R}^d)$-VALUED SOLUTION] *Assume b is continuous and there exists a finite constant $c > 0$ such that*

$$|b(u)| \leq c(1 + |u|) \tag{7.8}$$

holds for all $u \in \mathbb{R}$. Let $\eta \in C_{tem}(\mathbb{R}^d)$ and $\sigma(dtdx)$ satisfy condition (B) *with β_1, β_2. Then PDE (5.16) with initial condition η has a $C_{tem}(\mathbb{R}^d)$-valued solution which satisfies (7.2). If we additionally assume $\eta \in C_{tem}^+(\mathbb{R}^d)$ and $b(0) \geq 0$, then this solution is non-negative.*

Proof We may and do pick a sequence (b_n) of Lipschitz continuous functions approximating b uniformly on compacts. Also, the b_n can be chosen in such a manner that they fulfill (7.8) with a common constant c for all $n \geq 1$. By Theorem 7.3 there is for every

$n \geq 1$ a unique $C_{tem}(\mathbb{R})$-valued solutions u_n to PDE (5.16) with b replaced by b_n. Set $v_n := u_n - P_.\eta(.)$. As in Step 1 of the proof of Theorem 7.3 (or Step 1 of the proof of Theorem 6.8) one can verify that

$$\sup_{n\geq 1} \left| v_n(t,x) - v_n(t',x') \right| \ \leq \ c_{\lambda,T} \Big(|t-t'|^{\beta} + |x-x'|^{2\beta} \Big) e^{\lambda|x|}$$

holds for all $t,t' \leq T$, $x,x' \in \mathbb{R}$ with $|x-x'| \leq 1$ and $\lambda > 0$, where $\beta := \frac{\beta_1}{2} + \beta_2 - d/2$. In particular, $A := \{v_n : n \geq 1\}$ fulfills $(i)-(iii)$ of the the Arzelà-Ascoli type criterion in Proposition 3.12. Consequently, (v_n) is relatively compact in $C([0,\infty), C_{tem}(\mathbb{R}^d))$ w.r.t. $d_{tem,\infty}$. Let v denote the limit point of any convergent subsequence $(v_k) \subset (v_n)$ and set $u := P_.\eta(.) + v$. Taking (4.6) into account, u is easily seen to satisfy (7.2). We complete the proof of Theorem 7.5 by showing that u is a $C_{tem}(\mathbb{R}^d)$-valued solution to PDE (5.16).

Since $u, u_k \in C([0,\infty), C_{tem}(\mathbb{R}^d))$ and $d_{tem,\infty}(u,u_k) \to 0$, we obtain for all $\lambda, t > 0$:

$$\sup_{r\leq t} |u(r,.)|_{(-\lambda)} \leq c_{\lambda,t} < \infty, \qquad \sup_{k\geq 1}\sup_{r\leq t} |u_k(r,.)|_{(-\lambda)} \leq c_{\lambda,t} < \infty \tag{7.9}$$

where $|.|_{(-\lambda)}$ is defined as in Section 3.3. Let $\psi \in C_c^{\infty}(\mathbb{R}^d)$. From (7.9) we deduce

$$H_{t,\psi} := \Big(\sup_{k\geq 1}\sup_{r\leq t}\sup_{y\in \mathrm{supp}(\psi)} |u_k(r,y)| \Big) \vee \Big(\sup_{r\leq t}\sup_{y\in \mathrm{supp}(\psi)} |u(r,y)| \Big) < \infty.$$

Since $b(.)$ is uniformly continuous on $[-H_{t,\psi}, H_{t,\psi}]$ and $\lim_{k\to\infty} u_k(.,.) = u(.,.)$ holds uniformly on $[0,t] \times \mathrm{supp}(\psi)$, $\lim_{k\to\infty} b(u_k(.,.)) = b(u(.,.))$ holds uniformly on $[0,t] \times \mathrm{supp}(\psi)$, too. Also, $\lim_{k\to\infty} b_k(.) = b(.)$ holds uniformly on $[-H_{t,\psi}, H_{t,\psi}]$. So we obtain $\lim_{k\to\infty} b_k(u_k(.,.)) = b(u(.,.))$ uniformly on $[0,t] \times \mathrm{supp}(\psi)$. Therefore, the terms in (5.17) with u, b replaced by u_k, b_k converge to the corresponding terms in (5.17) as $k \to \infty$. Hence, u satisfies (5.17), i.e. u is a $C_{tem}(\mathbb{R}^d)$-valued solution to PDE (5.16).

If $\eta \in C_{tem}^{+}(\mathbb{R}^d)$ and $b(0) \geq 0$, then each u_k from the approximating sequence (u_k) of u can be chosen to be non-negative. Thus u is non-negative, too. This completes the proof of Theorem 7.5. $\square$

7.5 Refinement of the state space

So far the state space of solutions to PDE (5.16) was always be assumed to be $C_{tem}(\mathbb{R})$. Occasionally one wishes to work with a finer state space. In this section we show that a solution stays in $C_{rap}(\mathbb{R}^d)$ whenever the initial state η is an element of $C_{rap}(\mathbb{R}^d)$.

Theorem 7.6 [SOLUTION PRESERVING THE STATE SPACE $C_{rap}(\mathbb{R}^d)$] *Assume for every $T > 0$ there exists a finite constant $c_T > 0$ such that for all $t \leq T$, $x \in \mathbb{R}^d$ and $u \in \mathbb{R}$:*

$$|b(t,x,u)| \leq c_T |u|. \tag{7.10}$$

If $\sigma(dtdx)$ satisfies condition (B), *then every $C_{tem}(\mathbb{R}^d)$-valued solution to PDE (5.16) with initial condition $\eta \in C_{rap}(\mathbb{R}^d)$ is even $C_{rap}(\mathbb{R}^d)$-valued continuous.*

Proof Let u be a $C_{tem}(\mathbb{R})$-valued solution to PDE (5.16) with initial condition $\eta \in C_{rap}(\mathbb{R}^d)$. The continuity of u w.r.t. d_{tem} implies

$$\sup_{t\le T}\sup_{x\in\mathbb{R}^d} e^{-\lambda|x|}|u(t,x)| < \infty \tag{7.11}$$

for every $\lambda, T > 0$. By (7.10) we obtain for all $t \le T$ and $x \in \mathbb{R}^d$:

$$\begin{aligned}|u(t,x)| &\le |P_t\eta(x)| + \int_0^t \int_{\mathbb{R}^d} p_{t-r}(x,y)\,|b(r,y,u(r,y))|\,\sigma(drdy)\\ &\le |P_t\eta(x)| + \sup_{s\le t}\int_0^s \int_{\mathbb{R}^d} p_{s-r}(x,y)c_T|u(r,y)|\sigma(drdy).\end{aligned}$$

Hence, the Gronwall-type Lemma 4.14, (7.11) and Lemma 4.9 imply for every $\lambda, T > 0$:

$$\sup_{t\le T}\sup_{x\in\mathbb{R}^d} e^{\lambda|x|}|u(t,x)| < \infty. \tag{7.12}$$

Recall the definitions of Φ' and Φ'_2 from Section 7.2. With help of (7.12) we can show (as in Step 1 of the proof of Theorem 7.3, resp. Theorem 6.8) that

$$|\Phi'_2(u)(t,x) - \Phi'_2(u)(t',x')| \le c_{\lambda,T}\Big(|t-t'|^{\beta} + |x-x'|^{2\beta}\Big)e^{-\lambda|x|}e^{\lambda|x-x'|}$$

holds for all $t,t' \le T$, $x,x' \in \mathbb{R}^d$ and $\lambda > 0$, where $\beta := \frac{\beta_1}{2} + \beta_2 - d/2$. In particular, $(\Phi'(u)(t,.) : t \ge 0)$ is $C_{rap}(\mathbb{R}^d)$-valued continuous. By Lemma 4.9, $(P_t\eta(.) : t \ge 0)$ is $C_{rap}(\mathbb{R}^d)$-valued continuous, too. Hence,

$$u = \Phi'(u) = P_{.}\eta(.) + \Phi'_2(u)$$

is $C_{rap}(\mathbb{R}^d)$-valued continuous which was the claim of Theorem 7.6. $\square$

7.6 Solutions (drift to the inside)

In this section we assume b to be independent of t and x, i.e. we consider some continuous $b : \mathbb{R} \to \mathbb{R}$. Moreover, we suppose $b(0) = 0$ and $b(u) \le 0$ for all $u \ge 0$. For every $H > 0$, set $b_H(u) := [\{\inf_{v\in[0,H]} b(v)\} \vee b(u)]\mathbf{1}_{u\ge 0}$ for all $u \in \mathbb{R}$.

Theorem 7.7 [UNIQUE $C_b^+(\mathbb{R}^d)$-VALUED SOLUTIONS] *Suppose $b(0) = 0$ and $b(u) \le 0$ for all $u \ge 0$. Assume for every $H > 0$ there exists a finite constant $c_H > 0$ such that*

$$|b_H(u) - b_H(u')| \le c_H|u - u'|$$

holds for all $u, u' \in \mathbb{R}$. Let $\eta \in C_b^+(\mathbb{R}^d)$ and $\sigma(dtdx)$ satisfy condition (B) *with β_1, β_2. Then PDE (5.16) with initial condition η has a unique $C_b^+(\mathbb{R}^d)$-valued solution in the sense of Definition 5.21. Moreover, the solution satisfies (7.2).*

Proof First of all note that $0 \leq H_\eta := \sup_{t\geq 0}\sup_{x\in\mathbb{R}^d} P_t\eta(x) < \infty$ since $\eta \in C_b^+(\mathbb{R}^d)$. Since η and $b_{H_\eta}(.)$ satisfy the assumptions of Theorems 7.3 and 7.4, there exists a unique $C_{tem}^+(\mathbb{R}^d)$-valued solution $u(.,.)$ to PDE (5.16) with drift coefficient b_{H_η} and initial condition η. In particular, $u(.,.)$ satisfies (7.2). Since $u(.,.)$ is non-negative and the drift acts to the inside only, we have $0 \leq u(t,x) \leq H_\eta$ for all $t \geq 0$ and $x \in \mathbb{R}^d$. So $u(.,.)$ is also a solution to PDE (5.16) with drift coefficient b and initial condition η. In particular, $u(.,.)$ is $(C_b^+(\mathbb{R}^d), d_{tem})$-valued continuous and unique among $C_b^+(\mathbb{R}^d)$-valued solutions. □

7.7 Time reversal: backward equation

At the end of Chapter 7 we consider PDE (5.16) with reversed time. That is, for some fixed $t > 0$ we look for a real-valued function $u(.,t,.) = (u(s,t,x) : s \in [0,t], x \in \mathbb{R}^d)$ which satisfies the "final condition" $u(t,t,.) = \eta(.)$ and which behaves for decreasing s as a solution to PDE (5.16) behaves for increasing t. More precisely, we wish to obtain a function $u(.,t,.)$ which (formally) satisfies

$$\begin{aligned}
\frac{\partial}{\partial\tau}u(t-\tau,t,x) &= \frac{1}{2}\Delta u(t-\tau,t,x) + b(t-\tau,x,u(t-\tau,t,x))\frac{\sigma(dsdx)}{dsdx}(t-\tau,x) \\
u(t,t,x) &= \eta(x) \qquad \tau \in [0,t],\ x \in \mathbb{R}^d.
\end{aligned} \tag{7.13}$$

Equation (7.13) can clearly be written as

$$\begin{aligned}
-\frac{\partial}{\partial s}u(s,t,x) &= \frac{1}{2}\Delta u(s,t,x) + b(s,x,u(s,t,x))\frac{\sigma(dsdx)}{dsdx}(s,x) \\
u(t,t,x) &= \eta(x) \qquad s \in [0,t],\ x \in \mathbb{R}^d.
\end{aligned} \tag{7.14}$$

The precise meaning of this backward partial differential equation (BPDE) is not surprising. Heuristic calculations as in (5.11) suggest:

Definition 7.8 [SOLUTIONS TO BPDE] *A deterministic $\mathbb{F}$-valued continuous process $(u(s,t,.) : s \in [0,t])$ is said to be an $\mathbb{F}$-valued solution to BPDE (7.14) with final condition $\eta \in \mathbb{F}$ if for all $s \in [0,t]$ and $\psi \in C_c^\infty(\mathbb{R}^d)$:*

$$\langle u(s,t,.),\psi\rangle = \langle \eta,\psi\rangle + \int_s^t \langle u(r,t,.),\frac{1}{2}\Delta\psi\rangle dr + \int_s^t\int_{\mathbb{R}^d} b(r,y,u(r,t,y))\psi(y)\sigma(drdy). \tag{7.15}$$

A solution is said to be unique if any two solutions coincide pointwise.

Here the state space $\mathbb{F}$ of a solution $(u(s,t,.) : s \in [0,t])$ will again be one of the spaces $C_{tem}(\mathbb{R}^d)$, $C_b(\mathbb{R}^d)$ (both furnished with d_{tem}) or $C_{rap}(\mathbb{R}^d)$ (furnished with d_{rap}). As in the proof of Proposition 7.2 (or Proposition 6.7) one can show that a deterministic continuous $\mathbb{F}$-valued process $(u(s,t,.) : s \in [0,t])$ is a solution to BPDE (7.14) with final condition $u(t,t,.) = \eta \in \mathbb{F}$ if and only if it is a solution to the following integral equation:

$$u(s,t,x) = P_{t-s}\eta(x) + \int_s^t\int_{\mathbb{R}^d} p_{r-s}(x,y)b(r,y,u(r,t,y))\sigma(drdy), \quad s \in [0,t], x \in \mathbb{R}^d. \tag{7.16}$$

So it is not hard to verify the following remark on existence and uniqueness of solutions.

Remark 7.9 [EXISTENCE AND UNIQUENESS OF SOLUTION] *All results on existence and uniqueness of solutions to PDE (5.16) which were presented in Sections 7.2-7.6 remain true for BPDE (7.14) for every (fixed) $t > 0$.*

Proof Fix $t > 0$ and define $\sigma_t(dvdx)$ to be the unique Radon measure on $[0, \infty) \times \mathbb{R}^d$ satisfying $\sigma_t([0, r] \times B) = \sigma([t - r, t] \times B)$ for all $r \in [0, t]$ and $B \in \mathcal{B}(\mathbb{R}^d)$. By substituting $v := t - r$ we deduce from (7.16)

$$u(s, t, x) = P_{t-s}\eta(x) + \int_0^{t-s} \int_{\mathbb{R}^d} p_{t-v-s}(x, y) b(t - v, y, u(t - v, t, y)) \sigma_t(dvdy)$$

for all $s \in [0, t]$ and $x \in \mathbb{R}^d$. Setting $\tau := t - s$ we obtain

$$u(t - \tau, t, x) = P_{t-s}\eta(x) + \int_0^{\tau} \int_{\mathbb{R}^d} p_{\tau-v}(x, y) b(t - v, y, u(t - v, t, y)) \sigma_t(dvdy)$$

for all $\tau \in [0, t]$ and $x \in \mathbb{R}^d$. If we set $u_t(\tau, x) := u(t-\tau, t, x)$ and $b_t(v, y, u) := b(t-v, y, u)$, then this equation turns into

$$u_t(\tau, x) = P_{t-s}\eta(x) + \int_0^{\tau} \int_{\mathbb{R}^d} p_{\tau-v}(x, y) b_t(v, y, u_t(v, y)) \sigma_t(dvdy) \tag{7.17}$$

for all $\tau \in [0, t]$ and $x \in \mathbb{R}^d$. So it is easy to see that (7.16) has a (unique) $\mathbb{F}$-valued solution if and only if (7.17) has. Now, if $\sigma(dvdy)$ satisfies condition (B), then $\sigma_t(dvdy)$ satisfies condition (B) (restricted to the time interval $[0, t]$) as well. Also, all conditions on the drift coefficient $b = (b(v, y, u) : v \in [0, t], y \in \mathbb{R}^d, u \in \mathbb{R})$ considered so far do not depend on the "direction" of its time coordinate v. That means, if b and $\sigma(dvdy)$ satisfy the assumptions of any result of Sections 7.2-7.6, then b_t and $\sigma_t(dvdy)$ satisfy these assumptions, too. Hence, the claim follows from the fact that equation (7.17) is just a reformulation of PDE (5.16) with b and $\sigma(dvdy)$ replaced by b_t and $\sigma_t(dvdy)$ (recall Proposition 7.2). □

An intrinsic feature of the backward formulation is the following. If BPDE (7.14) has a unique $\mathbb{F}$-valued solution for every $t \geq 0$ and $\eta \in \mathbb{F}$, then these solutions form an (inhomogeneous) semigroup on $\mathbb{F}$:

Lemma 7.10 [SEMIGROUP PROPERTY] *If BPDE (7.14) has a unique $\mathbb{F}$-valued solution $(U_{s,t}\eta(.) : s \in [0, t])$ for every $t \geq 0$ and $\eta \in \mathbb{F}$, then $(U_{s,t})_{0 \leq s \leq t}$ provides an (inhomogeneous) semigroup on $\mathbb{F}$, i.e. $U_{t,t} = \mathbb{I}$ and $U_{s,t} = U_{s,v}U_{v,t}$ hold on $\mathbb{F}$ for all $0 \leq s \leq v \leq t$.*

Note that the semigroup property does not hold in general for solutions of the (forward) PDE (5.16) since the drift term is not required to be homogeneous in time.

Proof (of Lemma 7.10) Fix $v, t \geq 0$ such that $0 \leq v < t$. We intend to show $U_{s,v}U_{v,t} \equiv U_{s,t}$ for all $s \in [0, v]$. Set $U'_{s,t} := \mathbf{1}_{[0,v]}(s) U_{s,v} U_{v,t} + \mathbf{1}_{(v,t]}(s) U_{s,t}$ for all $s \in [0, t]$. Note that $U'_{s,t}$ is $\mathbb{F}$-valued continuous in s. By assumption we have for all $s \in [0, v]$, $x \in \mathbb{R}^d$ and $\eta \in \mathbb{F}$

$$U'_{s,t}\eta(x) \;=\; U_{s,v}U_{v,t}\eta(x) \tag{7.18}$$

$$
\begin{aligned}
&= P_{v-s}(U_{v,t}\eta)(x) + \int_s^v \int_{\mathbb{R}^d} p_{r-s}(x,y) b(r,y,U_{r,v}U_{v,t}\eta(y))\sigma(drdy) \\
&= P_{v-s}\Big(P_{t-v}\eta(.) + \int_v^t \int_{\mathbb{R}^d} p_{r-v}(.,y) b(r,y,U_{r,t}\eta(y))\sigma(drdy)\Big)(x) \\
&\qquad\qquad\qquad + \int_s^v \int_{\mathbb{R}^d} p_{r-s}(x,y) b(r,y,U_{r,v}U_{v,t}\eta(y))\sigma(drdy) \\
&= P_{t-s}\eta(x) + \int_v^t \int_{\mathbb{R}^d} p_{v-s+r-v}(x,y) b(r,y,U_{r,t}\eta(y))\sigma(drdy) \\
&\qquad\qquad\qquad + \int_s^v \int_{\mathbb{R}^d} p_{r-s}(x,y) b(r,y,U_{r,v}U_{v,t}\eta(y))\sigma(drdy) \\
&= P_{t-s}\eta(x) + \\
&\quad \int_s^t \int_{\mathbb{R}^d} p_{r-s}(x,y)\Big[\mathbf{1}_{[s,v]}(r) b(r,y,U_{r,v}U_{v,t}\eta(y)) + \mathbf{1}_{(v,t]}(r) b(r,y,U_{r,t}\eta(y))\Big]\sigma(drdy) \\
&= P_{t-s}\eta(x) + \int_s^t \int_{\mathbb{R}^d} p_{r-s}(x,y) b(r,y,U'_{r,t}\eta(y))\sigma(drdy).
\end{aligned}
$$

Also, for all $s \in (v,t]$, $x \in \mathbb{R}^d$ and $\eta \in \mathbb{F}$,

$$
\begin{aligned}
U'_{s,t}\eta(x) \;=\; U_{s,t}\eta(x) \quad &= \quad P_{t-s}\eta(x) + \int_s^t \int_{\mathbb{R}^d} p_{r-s}(x,y) b(r,y,U_{r,t}\eta(y))\sigma(drdy) \\
&= \quad P_{t-s}\eta(x) + \int_s^t \int_{\mathbb{R}^d} p_{r-s}(x,y) b(r,y,U'_{r,t}\eta(y))\sigma(drdy).
\end{aligned}
$$

That is, $(U'_{s,t}\eta(x) : s \in [0,t], x \in \mathbb{R}^d)$ is an $\mathbb{F}$-valued solution to BPDE (7.14) with final condition η. However, we assumed the solution to be unique and so $U'_{s,t}\eta = U_{s,t}\eta$ holds for all $s \in [0,t]$. Then the claim follows from the first line in (7.18). □

8 Media for Brownian particles

The goal of this chapter is to model a medium for a Brownian particle. Here *medium* means the following: A clock is attached to the particle and the running of the "time" is governed by the collision of the particle with the medium. In particular, the time scale of the particle differs from the time scale of the attached clock. The medium influences the particle only through the clock, i.e. the clock plays the crucial role in modelling media. We want the "time" of the clock to run proportionally to the collision of the particle with the medium, i.e. the more intense the collision the faster should the "time" run.

8.1 A first example: singletons as media

Let $B = (B_t : t \geq 0)$ be a Brownian motion in $\mathbb{R}$. If the medium is wanted to be a singleton $x \in \mathbb{R}$, then the *Brownian local time* L_x of B at level x is a good choice for the clock. The local time L_x, first considered by Lévy in 1948 ([Lév48]), can be defined by

$$L_x(t) := \lim_{\epsilon \downarrow 0} L_x^\epsilon(t) \qquad \text{where } L_x^\epsilon(t) := \int_0^t \frac{1}{2\epsilon} \mathbf{1}_{[x-\epsilon, x+\epsilon]}(B_r)dr, \qquad t \geq 0. \tag{8.1}$$

Informally, the process L_x^ϵ measures the amount of time spent by the Brownian motion B in the ϵ-hull around x, whereas L_x displays the amount of time spent by B at level x. In fact, L_x does not vary when B is doing an excursion away from x, and increases continuously otherwise. Hence L_x is exactly doing what it was wished to do, and so L_x appoints the singleton x *medium*. Note that the restriction to dimension one is essential for obtaining a non-degenerated limit in (8.1). It should also be mentioned that $(L_x(t) : t \geq 0, x \in \mathbb{R})$ forms a jointly continuous field. For a discussion of the Brownian local time see [Kni81].

There are actually several ways to define and to construct local times. Let us mention four types of processes for which the local time at a fixed level can be constructed (in different ways). Brownian motion is an example for each of these types.

Continuous semimartingales. The key for the construction of the local time is Itô's formula; see standard references as [IW89], [KS91], [Kal97], [RY98] etc.

Continuous Gaussian processes. The existence of the local time depends on the covariation function Γ. Informally, $\Gamma(t, t')$ should be "small" (whereby the second moment of the process' increment on the interval $[t, t']$ is "large") when $t' - t$ is "small". This indicates a "strong" fluctuation of the process. See [GH80] or [Adl81].

Regenerative processes. Regenerative processes are certain cadlag (Lévy-, Markov-, ...) processes for which excursion theory exists. Excursion theory is the key for the construction of the local time; see, for instance, [Ber96] or [Kal97].

Feller-Markov processes. The local time can be constructed as a continuous additive functional (in the sense of (3.23)) of the process; see [Vol60], [BG68], [Kal97] etc.

In the remainder of this chapter we shall turn back to the latter approach. However, we construct local times (and, more generally, collision processes) as continuous additive functionals in the sense of Definition 3.44 rather than in the sense of (3.23).

In conclusion we would like to mention that the Brownian local time in $\mathbb{R}$ could also be defined as follows

$$L_x(t) := \lim_{\epsilon \downarrow 0} \hat{L}_c^\epsilon(t) \qquad \text{where } \hat{L}_x^\epsilon(t) := \int_0^t p_\epsilon(x, B_r)dr, \qquad t \geq 0, \tag{8.2}$$

i.e. we may replace the weighted indicator function by the heat kernel.

8.2 Space measures as media

In the last section we have seen how to regard a singleton as a medium for a Brownian particle. In the present section we would like to regard a space measure as a medium. Let $B = (B_t : t \geq 0)$ be a Brownian motion in $\mathbb{R}^d$ and $\mu(dx)$ be a Borel measure on $\mathbb{R}^d$. For the moment assume $d = 1$. If $\mu(dx)$ is wanted to be considered as a medium for B, then B has to be equipped with a suitable clock. It is quite plausible that the *collision local time* $L_{[B,\mu]}$ of B and $\mu(dx)$, given by

$$\begin{aligned} L_{[B,\mu]}(t) \quad &:= \quad \int_{\mathbb{R}} L_x(t)\mu(dx) \qquad (8.3)\\ \text{“} = \text{”} \quad & \int_{\mathbb{R}} \Big(\text{Brownian local time at level } x \text{ at } t\Big)\Big(\text{medium's mass at } x\Big)dx, \end{aligned}$$

is a proper choice for the clock; the Brownian local time L_x at level x was defined in (8.1). The definition in (8.3) makes sense since we assumed $d = 1$ (we already mentioned that L_x does only exist in dimension one). However, if we replace in (8.3) $L_x(t)$ by $\hat{L}_x^\epsilon$ from (8.2) and let $\epsilon \downarrow 0$, then the definition could make sense even in higher dimensions. Let us write this definition down for arbitrary $d \geq 1$. The *collision local time* of a d-dimensional Brownian motion B and the Borel measure $\mu(dx)$ is defined by

$$L_{[B,\varrho]}(t) := \lim_{\epsilon \downarrow 0} \int_{\mathbb{R}^d} \int_0^t p_\epsilon(x, B_r)dr\mu(dx), \qquad t \geq 0. \tag{8.4}$$

Of course, $\mu(dx)$ has to be ruled out to be too singular for $d \geq 2$. It should be only moderately singular in a suitable sense (which has to be specified yet; cf. the next section). Otherwise the limit in (8.4) could be degenerated.

With $L_{[B,\mu]}$ we have found a "clock" which measures the collision of B and a (moderately singular or non-singular) space measure $\mu(dx)$ properly. Now the question arises how to measure the collision of B and a time-space measure $\mu(dtdx) \in \mathcal{M}([0,\infty) \times \mathbb{R}^d)$. An answer will be given in the next section. In particular, we shall introduce the notion of collision measures of two Borel measures on $[0,\infty) \times \mathbb{R}^d$.

8.3 Collision measures and collision processes

Let $d \geq 1$ and consider two time-space measures $\mu(dtdx), \mu'(dtdx) \in \mathcal{M}([0,\infty) \times \mathbb{R}^d)$. Our intension here is to define a measure $C_{[\mu,\mu']}(dtdx) \in \mathcal{M}([0,\infty) \times \mathbb{R}^d)$ that measures the "collision" of mass of μ with mass of μ'. If μ and μ' possess Lebesgue densities f, respectively f', then a plausible definition of $C_{[\mu,\mu']}$ is

$$C_{[\mu,\mu']}(dtdx) := f(t,x)f'(t,x)dtdx. \tag{8.5}$$

In other words, the local mass of $C_{[\mu,\mu']}$ is determined by the product of the local masses of μ and μ'. Now assume that at least one of the measures $\mu(dtdx)$ and $\mu'(dtdx)$ is singular w.r.t. the Lebesgue measure $dtdx$. Then the definition in (8.5) cannot be used any more. The way out is the following approximation of $C_{[\mu,\mu']}$. We say $C_{[\mu,\mu']} \in \mathcal{M}([0,\infty)\times\mathbb{R}^d)$ is the *collision measure* of μ and μ' if for every $\psi \in C_c([0,\infty)\times\mathbb{R}^d)$ the following equation holds

$$\begin{aligned}&\int_0^\infty \int_{\mathbb{R}^d} \psi(t,x) C_{[\mu,\mu']}(dtdx) \qquad (8.6)\\ &\quad = \lim_{\epsilon\downarrow 0} \int_0^\infty \int_{\mathbb{R}^d} \int_0^\infty \int_{\mathbb{R}^d} \psi\Big(\frac{t+t'}{2}, \frac{x+x'}{2}\Big) p_\epsilon(t,t') p_\epsilon(x,x') \mu'(dt'dx')\mu(dtdx).\end{aligned}$$

Note that we may replace $\psi(\frac{t+t'}{2}, \frac{x+x'}{2})$ by $\psi(t,x)$ in (8.6) without altering the definition because of the uniform continuity of the test function ψ on compacts. The definition of $C_{[\mu,\mu']}$ through (8.6) is a generalization of the one in (8.5). Indeed, if μ and μ' possess Lebesgue densities f, respectively f', then the definitions coincide. If one of the measures $\mu(dtdx) = \mu_1(t,dx)\mu_2(dt)$ and $\mu'(dtdx) = \mu_1'(t,dx)\mu_2'(dt)$, say $\mu'(dtdx)$, is Lebesgue in time, i.e. $\mu_2'(dt) = dt$, then we use a little different definition of $C_{[\mu,\mu']}(dtdx)$. Instead of (8.6) we require

$$\int_0^\infty \int_{\mathbb{R}^d} \psi(t,x) C_{[\mu,\mu']}(dtdx) = \lim_{\epsilon\downarrow 0} \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \psi(t,x) p_\epsilon(x,x') \mu_1'(t,dx')\mu(dtdx). \qquad (8.7)$$

On the one hand, defining the collision measure through (8.7) simplifies the calculations, that will be done in the sequel, essentially. On the other hand, $\mu_2'(dt) = dt$ does not automatically imply that (8.6) and (8.7) lead to the same collision measure. However, this should be the case if $\mu_2(dt)$ is not too singular. A rigorous criterium for the equivalence of (8.6) and (8.7) seems to be an interesting problem on its own. The definition of the collision measure can easily be extended to random measures. If μ or μ' (and so $C_{[\mu,\mu']}$) are random measures on $[0,\infty)\times\mathbb{R}^d$, then the convergence in (8.7) has to be replaced by some suitable stochastic convergence (e.g. L^2- or a.s.-convergence). If μ and μ' represent measure-valued processes, i.e. if $\mu_2'(dt) = \mu_2'(dt) = dt$, then the definition of the collision measure through (8.7) coincides with the definition of the collision local time used by Barlow, Evans and Perkins ([BEP91], Section 1).

The notion of collision measure can be used to redefine the collision local time $L_{[B,\mu]}$ of a Brownian motion and a space measure $\mu(dx)$ introduced in (8.4). Indeed: Let B be a d-dimensional Brownian motion. We may and do choose B as a canonical continuous Markov process $B = [B, \mathbb{P}_{s,\nu} : s \geq 0, \nu \in \mathcal{M}_1(\mathbb{R}^d)]$; hence $\Omega = C([0,\infty),\mathbb{R}^d)$ and $B = \bar{\pi}$. Moreover, B is known to satisfy (3.19) whereby we may switch to the usual augmentation of the natural filtration (recall Proposition 3.39). If we think of the Brownian motion B as the random measure $\delta_{B_t}(dx)dt$ on $[0,\infty)\times\mathbb{R}^d$, then the collision local time $L_{[B,\mu]}(t)$ of B and $\mu(dx)$ at time t is nothing but the collision measure of $\delta_{B_t}(dx)dt$ and $\mu(dx)dt$ evaluated at $[0,t]\times\mathbb{R}^d$. That is,

$$L_{[B,\mu]}(t) = C_{[\delta_{B_t}(dx)dt, \mu(dx)dt]}([0,t]\times\mathbb{R}^d) \qquad \forall t \geq 0.$$

More generally, a suitable benchmark for the collision of a Brownian particle B with a time-space measure $\mu(dtdx)$ up to time t is the collision measure of $\delta_{B_t}(dx)dt$ and $\mu(dtdx)$ – in the sense of (8.7) – evaluated at $[0,t]\times\mathbb{R}^d$. We will call the corresponding process *collision process* of B and $\mu(dtdx)$ and denote it by $C_{[B,\mu]}$.[24] That is,

$$C_{[B,\mu]}(t) := C_{[\delta_{B_t}(dx)dt,\mu(dtdx)]}([0,t]\times\mathbb{R}^d) \qquad \forall t\geq 0. \tag{8.8}$$

In fact we use a definition which has a bit weaker demands on $C_{[B,\mu]}$. In Theorem 8.2 it will be apparent that the definition nevertheless corresponds to (8.8) (cf. (8.10)).

Definition 8.1 [CONTINUOUS COLLISION PROCESS] *Let $\mu(dtdx)\in\mathcal{M}([0,\infty)\times\mathbb{R}^d)$. Then a CAF $C_{[B,\mu]}$ of the Brownian motion B with characteristic h of the form*

$$h_{s,t}(x) = \int_s^t \int_{\mathbb{R}^d} p_{r-s}(x,y)\mu(drdy), \qquad 0\leq s\leq t, x\in\mathbb{R}^d$$

is called continuous collision process (CCP) of B and $\mu(dtdx)$.

While uniqueness (up to indistinguishability) of the CCP is ensured by the definition (recall Proposition 3.45), existence might fail. So it is natural to ask for which measures $\mu(dtdx)\in\mathcal{M}([0,\infty)\times\mathbb{R}^d)$ and in which dimensions $d\geq 1$ does the CCP exist. Intuitively, the closed support of $\mu(dtdx)$ should be large in a suitable sense. Otherwise the Brownian motion would not "meet" μ's mass. In Section 2.3 we mentioned that a set must be large in some sense if its Hausdorff dimension is large; and in Section 2.4 we gave a tool for determining a lower bound for the Hausdorff dimension of the closed support of a Borel measure (cf. Proposition 2.8). Having this in mind, the form of the following condition for the existence of the CCP is not surprising. Condition (B) was defined in Definition 2.22.

Theorem 8.2 [EXISTENCE AND APPROXIMATION OF CCP] *Pick $\mu(dtdx)\in\mathcal{M}([0,\infty)\times\mathbb{R}^d)$ and define for every $\epsilon>0$ and all $t\geq 0$ and $f\in C([0,\infty),\mathbb{R}^d)$:* [25]

$$C^\epsilon_{[B,\mu]}(t,f) := \int_0^t \int_{\mathbb{R}^d} p_\epsilon(y,f(r))\mu(drdy). \tag{8.9}$$

If $\mu(dtdx)$ satisfies condition (B)*, then the CCP $C_{[B,\mu]}$ of the Brownian motion B and $\mu(dtdx)$ exists and we have for every $0\leq s\leq T$ and $x\in\mathbb{R}^d$:*

$$\lim_{\epsilon\downarrow 0} \mathbb{E}_{s,x}\Big[\sup_{t\in[s,T]}\Big|C^\epsilon_{[B,\mu]}(s,t] - C_{[B,\mu]}(s,t]\Big|^2\Big] = 0. \tag{8.10}$$

Moreover, we obtain for every $0\leq s\leq t$, $x\in\mathbb{R}^d$ and $g\in C_b^+([0,\infty)\times\mathbb{R}^d)$:

$$\mathbb{E}_{s,x}\Big[\int_s^t g(r,B_r)dC_{[B,\mu]}(r)\Big] = \int_s^t\int_{\mathbb{R}^d} p_{r-s}(x,y)g(r,y)\mu(drdy). \tag{8.11}$$

[24]Note that, if $\mu(dtdx)=\mu_1(dx)dt$, then $C_{[B,\mu]} = L_{[B,\mu_1]}$.

[25]Recall that $C([0,\infty),\mathbb{R}^d)$ plays the role of Ω.

The proof of Theorem 8.2 is postponed to Section 8.5. For a brief discussion of similar results known from literature see Section 8.6 below. Examples for measures satisfying condition (B) have been presented in Section 2.8. In conclusion we would like to mention that the characteristic h of the CCP can easily be shown to satisfy

$$h_{s,t}(x) \leq c_T |s-t|^{\beta_1/2+\beta_2-d/2} \qquad \forall s,t \in [0,T] \quad \text{(uniformly in } x \in \mathbb{R}^d) \tag{8.12}$$

for every $T > 0$. Indeed, (8.12) follows from (8.11) and Lemmas 4.2(i) $\Rightarrow$ (ii) and 4.4(i).

8.4 Time-space measures as media

In Section 8.2 we have seen that a space measure $\mu(dx)$ can be regarded as a medium for a Brownian particle via the collision local time of the Brownian motion and $\mu(dx)$. In the last section we generalized the notion of collision local times by introducing the CCP of the Brownian motion and a time-space measure. So we are now in the position to regard a time-space measure $\mu(dtdx)$ as a *medium* for a Brownian particle. In fact, $\mu(dtdx)$ becomes a medium for a Brownian particle B if the particle is furnished with the "clock" $C_{[B,\mu]}$. Here $C_{[B,\mu]}$ is the CCP of B and μ from Definition 8.1. Theorem 8.2 hence provides a quite general class of media for Brownian particles, namely the class of measures $\mu(dtdx)$ satisfying condition (B). In Section 9.3 such media will be used for governing the branching times of branching Brownian particles.

8.5 Construction of continuous collision process

This section is devoted to the proof of Theorem 8.2. Let $\mu(dtdx) \in \mathcal{M}([0,\infty) \times \mathbb{R}^d)$ satisfy condition (B) and $B = [B, \mathbb{P}_{s,\nu} : s \geq 0, \nu \in \mathcal{M}_1(\mathbb{R}^d)]$ be a canonical continuous d-dimensional Brownian motion (i.e. $\Omega = C([0,\infty), \mathbb{R}^d)$ and $B = \bar{\pi}$). In order to keep the following calculations as clear as possible we use the symbol A^ϵ instead of $C^\epsilon_{[B,\mu]}$ (which was defined in (8.9)). As the first step we show that A^ϵ is continuous for every $\epsilon > 0$.

Lemma 8.3 *For every $\epsilon > 0$ and $f \in C([0,\infty), \mathbb{R}^d)$, the function $(A^\epsilon(t,f) : t \geq 0)$ is continuous.*

Proof Let $\epsilon > 0$. By condition (B), Lemma 4.2(i) $\Rightarrow$ (ii) and Lemma 4.4(i) we obtain

$$\begin{aligned}
&|A^\epsilon(t,f) - A^\epsilon(t',f)| \\
&= \int_t^{t'} \int_{\mathbb{R}^d} p_\epsilon(y, f(r)) \mu(drdy) \;\leq\; \int_t^{t'} \int_{\mathbb{R}^d} \frac{1}{(2\pi\epsilon)^{d/2}} e^{-\frac{|y-f(r)|^2}{2\epsilon}} \mu_1(r,dy)\mu_2(dr) \\
&\leq \int_t^{t'} \int_{\mathbb{R}^d} \frac{1}{(2\pi\epsilon)^{d/2}}\, c\, \epsilon^{\beta_1/2} \mu_2(dr) \;\leq\; c_\epsilon \int_t^{t'} \mu_2(dr) \;\leq\; c_{\epsilon,T}\, |t-t'|^{\beta_2}
\end{aligned}$$

for all $0 \leq t \leq t' \leq T$ and $f \in C([0,\infty), \mathbb{R}^d)$. □

For the proof of the next lemma we need the following elementary fact. If $\kappa \in \mathcal{M}([0,\infty))$ and $\phi : [0,\infty) \to \mathbb{R}$ is locally κ-integrable, then we have for all $0 \leq s \leq t$:

$$\Big(\int_s^t \phi(r)\kappa(dr) \Big)^2 = 2 \int_s^t \int_r^t \phi(r)\phi(v)\kappa(dv)\kappa(dr). \tag{8.13}$$

Recall our convention $A(s,t] = \int_s^t dA(r)$ for any $\mathbb{R}_+$-valued non-decreasing continuous A.

Lemma 8.4 *For every* $0 \le s < T$ *and* $\nu \in \mathcal{M}_1(\mathbb{R}^d)$ *we have:*

$$\lim_{\epsilon,\epsilon'\downarrow 0} \mathbb{E}_{s,\nu}\Big[\sup_{t\in[s,T]} |A^\epsilon(s,t] - A^{\epsilon'}(s,t]|^2\Big] = 0.$$

Proof W.l.o.g. we assume $s = 0$. We may also restrict to measures $\nu = \delta_x$, $x \in \mathbb{R}^d$. In fact, (8.15) and (8.16) remain true for general $\nu \in \mathcal{M}_1(\mathbb{R}^d)$ since $\mathbb{P}_{s,\nu} = \int \mathbb{P}_{s,x}\nu(dx)$. The process A^ϵ is easily seen to be a CAF of B in the sense of Definition 3.44 (continuity was obtained in Lemma 8.3). Its characteristic is

$$h^\epsilon_{s,t}(x) = \int_s^t \int_{\mathbb{R}^d} p_{r-s+\epsilon}(x,y)\mu(drdy), \qquad 0 \le s \le t, x \in \mathbb{R}^d.$$

Hence, by Lemma 3.47,

$$M^\epsilon_t := h^\epsilon_{t,T}(B_t) + A^\epsilon(0,t], \qquad t \in [0,T] \tag{8.14}$$

is an $(\bar{\mathcal{F}}_t^{B,\mathbb{P}_{0,x}})$-martingale under $\mathbb{P}_{0,x}$. With help of Doob's inequality (cf. Proposition 3.20), equation (8.13), the Markov property of B, Lemma 4.8, Lemma 4.2$(i) \Rightarrow (ii)$ and Lemma 4.4(i) we obtain for arbitrary $\theta \in (0, \beta_1/2 + \beta_2 - d/2)$:

$$\begin{aligned}
&\mathbb{E}_{0,x}\Big[\sup_{t\le T} |M^\epsilon(t) - M^{\epsilon'}(t)|^2\Big] \qquad (8.15)\\
&\le 4\,\mathbb{E}_{0,x}\Big[|M^\epsilon(T) - M^{\epsilon'}(T)|^2\Big] = 4\,\mathbb{E}_{0,x}\Big[|A^\epsilon(T) - A^{\epsilon'}(T)|^2\Big]\\
&= 4\,\mathbb{E}_{0,x}\Big[\Big(\int_0^T \Big\{\int_{\mathbb{R}^d} (p_\epsilon(y,B_r) - p_{\epsilon'}(y,B_r))\mu_1(r,dy)\Big\}\mu_2(dr)\Big)^2\Big]\\
&= 4\,\mathbb{E}_{0,x}\Big[2\int_0^T \int_r^T \Big\{\int_{\mathbb{R}^d} \Big(p_\epsilon(y,B_r) - p_{\epsilon'}(y,B_r)\Big)\mu_1(r,dy)\\
&\qquad\times \int_{\mathbb{R}^d} \Big(p_\epsilon(y,B_v) - p_{\epsilon'}(y,B_v)\Big)\mu_1(v,dy)\Big\}\mu_2(dv)\mu_2(dr)\Big]\\
&\le 8\,\mathbb{E}_{0,x}\Big[\int_0^T \int_r^T \Big\{\int_{\mathbb{R}^d} \Big(p_\epsilon(y,B_r) + p_{\epsilon'}(y,B_r)\Big)\mu_1(r,dy)\\
&\qquad\times \mathbb{E}_{r,B_r}\Big[\int_{\mathbb{R}^d} \Big(p_\epsilon(y,B_v) - p_{\epsilon'}(y,B_v)\Big)\mu_1(v,dy)\Big]\Big\}\mu_2(dv)\mu_2(dr)\Big]\\
&\le 8\,\mathbb{E}_{0,x}\Big[\int_0^T \Big\{\int_{\mathbb{R}^d} \Big(p_\epsilon(y,B_r) + p_{\epsilon'}(y,B_r)\Big)\mu_1(r,dy)\\
&\qquad\times \int_r^T \int_{\mathbb{R}^d} |p_{v-r+\epsilon}(y,B_r) - p_{v-r+\epsilon'}(y,B_r)|\,\mu_1(v,dy)\mu_2(dv)\Big\}\mu_2(dr)\Big]\\
&\le 8\,\mathbb{E}_{0,x}\Big[\int_0^T \int_{\mathbb{R}^d} \Big(p_\epsilon(y,B_r) + p_{\epsilon'}(y,B_r)\Big)\mu_1(r,dy)\, c_{T,\theta}\,|\epsilon-\epsilon'|^\theta \mu_2(dr)\Big]\\
&\le c'_{T,\theta}\,|\epsilon-\epsilon'|^\theta \int_0^T \int_{\mathbb{R}^d} \Big(p_{r+\epsilon}(y,x) + p_{r+\epsilon'}(y,x)\Big)\mu(drdy) \;\le\; c''_{T,\theta}\,|\epsilon-\epsilon'|^\theta.
\end{aligned}$$

By another application of Lemma 4.8 we also get

$$\begin{aligned}
&\mathbb{E}_{0,x}\Big[\sup_{t\leq T}|h^{\epsilon}_{t,T}(B_t)-h^{\epsilon'}_{t,T}(B_t)|^2\Big] &(8.16)\\
&= \mathbb{E}_{0,x}\Big[\sup_{t\leq T}\Big|\int_t^T\int_{\mathbb{R}^d}\Big(p_{r-t+\epsilon}(y,B_t)-p_{r-t+\epsilon'}(y,B_t)\Big)\mu(drdy)\Big|^2\Big]\\
&\leq \mathbb{E}_{0,x}\Big[\sup_{t\leq T}\ c_{T,\theta}\ |\epsilon-\epsilon'|^{\theta}\Big] \ = \ c_{T,\theta}\ |\epsilon-\epsilon'|^{\theta}.
\end{aligned}$$

Then the claim of the Lemma follows from (8.14), (8.15) and (8.16). □

Now, consider arbitrary $s\geq 0$ and $\nu\in\mathcal{M}_1(\mathbb{R}^d)$. Because of Lemma 8.4 we can find with help of Cantor's diagonal argument a sequence $(\epsilon_n)\subset(0,1]$ such that $\epsilon_n\downarrow 0$ and, for every $T=1,2,\ldots$, we have for all $n\geq T$:

$$\mathbb{E}_{s,\nu}\Big[\sup_{t\in[s,T]}|A^{\epsilon_k}(s,t]-A^{\epsilon_{k'}}(s,t]|^2\Big]\leq 2^{-n}\qquad\forall k,k'\geq n. \tag{8.17}$$

Recall $\Omega=C([0,\infty),\mathbb{R}^d)$ and set

$$A^{(s,\nu)}(t,f):=\begin{cases}\limsup_{n\to\infty}A^{\epsilon_n}(t,f) & , \quad (t,f)\in N^c\\ 0 & , \quad (t,f)\in N\end{cases}$$

for all $t\geq s$ and $f\in\Omega$, where $N:=\{(t,f)\in[s,\infty)\times\Omega:\ \limsup_{n\to\infty}A^{\epsilon_n}(t,f)=\infty\}$; be aware that the sequence (ϵ_n) depends on s and ν. With help of Fatou's lemma and (8.17) it is easy to show that

$$\lim_{\epsilon\downarrow 0}\ \mathbb{E}_{s,\nu}\Big[\sup_{t\in[s,T]}|A^{\epsilon}(s,t]-A^{(s,\nu)}(s,t]|^2\Big]=0\qquad\forall T>s. \tag{8.18}$$

In particular, taking Lemma 8.3 into account,

$$(A^{(s,\nu)}(s,t]:t>s)\text{ is }\mathbb{P}_{s,\nu}\text{-almost surely continuous.} \tag{8.19}$$

Moreover, we clearly have for every $\epsilon>0$:

$$f\mapsto A^{\epsilon}((u,v],f)\text{ is }[\tilde{\mathcal{F}}^B_{(s,t)},\mathcal{B}(\mathbb{R}_+)]\text{-measurable}\qquad\forall t>v>u>s. \tag{8.20}$$

Note that (8.18), (8.19) and (8.20) hold for all $s\geq 0$ and $\nu\in\mathcal{M}_1(\mathbb{R}^d)$, and recall that A^{ϵ} is non-decreasing and continuous. So we can apply Lemma 2.4.2 of [Dyn94] which yields a "nice" functional A which does not depend on s and ν. More precisely, there exists a functional $A:(t,f)\mapsto A(t,f)$, $[0,\infty)\times\Omega\to\mathbb{R}_+$ satisfying (i), (ii) and (iv) of Definition 3.44 as well as

$$\lim_{\epsilon\downarrow 0}\mathbb{E}_{s,\nu}\Big[|A^{\epsilon}(s,t]-A(s,t]|\Big]=0\qquad\forall t>s \tag{8.21}$$

for every $s\geq 0$ and $\nu\in\mathcal{M}_1(\mathbb{R}^d)$. With help of (8.18), (8.21) and Cantor's diagonal argument we can find, for every s and ν, a sequence $(\epsilon_n)\subset(0,1]$ such that $\epsilon_n\downarrow 0$ and $\mathbb{P}_{s,\nu}$-almost surely: $\lim_{n\to\infty}A^{\epsilon_n}(t,.)=A^{(s,\nu)}(t,.)$ and $\lim_{n\to\infty}A^{\epsilon_n}(t,.)=A(t,.)$ for all rational $t > s$. Thus we have $\mathbb{P}_{s,\nu}$-almost surely: $A^{(s,\nu)}(t,.) = A^{(s,\nu)}(t,.)$ for all rational $t > s$. Since both $(A(s,t] : t > s)$ and $(A^{(s,\nu)}(s,t] : t > s)$ are non-decreasing and the latter is also $\mathbb{P}_{s,\nu}$-almost surely continuous, we obtain

$$(A(s,t] : t > s) \text{ and } (A^{(s,\nu)}(s,t] : t > s) \text{ are } \mathbb{P}_{s,\nu}\text{-indistinguishable} \tag{8.22}$$

for every $s \geq 0$ and $\nu \in \mathcal{M}_1(\mathbb{R}^d)$. In particular, A also satisfies (iii) of Definition 3.44. Hence, A is a CAF of B. To complete the proof of Theorem 8.2 it suffices to show that moment formula (8.11) holds for $C_{[B,\mu]} := A$; setting $g := \mathbf{1}$ in (8.11) we see that A has the required characteristic in order to be the CCP of B and $\mu(dtdx)$.

Lemma 8.5 *Moment formula (8.11) holds for* $C_{[B,\mu]} := A$.

Proof Fix $s \geq 0$ and $x \in \mathbb{R}^d$. Recall that A^ϵ as well as A are continuous non-decreasing processes. By (8.18) and (8.22) there exists a sequence $(\epsilon_n) \subset (0,1]$ such that $\epsilon_n \downarrow 0$ and $A^{\epsilon_n}(dr)|_{[s,t]}$ converges weakly to $A(dr)|_{[s,t]}$ in $\mathcal{M}_f([s,t])$ (as $n \to \infty$) $\mathbb{P}_{s,x}$-almost surely, for every $t \geq s$. Thus we obtain for every $t \geq s$ and $g \in C_b^+([0,\infty) \times \mathbb{R}^d)$:

$$\int_s^t g(r,B_r)dA(r) = \lim_{n\to\infty} \int_s^t g(r,B_r)dA^{\epsilon_n}(r) \qquad \mathbb{P}_{s,x}\text{-almost surely.} \tag{8.23}$$

Proceeding as for the estimate in (8.15) we also obtain

$$\begin{aligned}
&\sup_{n\geq 1} \mathbb{E}_{s,x}\Big[\Big(\int_s^t g(r,B_r)dA^{\epsilon_n}(r)\Big)^2\Big] \qquad\qquad (8.24)\\
&\leq \|g\|_\infty^2 \sup_{n\geq 1} \mathbb{E}_{s,x}\Big[\Big(\int_s^t dA^{\epsilon_n}(r)\Big)^2\Big] = \|g\|_\infty^2 \sup_{n\geq 1} \mathbb{E}_{s,x}\Big[\Big(A^{\epsilon_n}(t) - A^{\epsilon_n}(s)\Big)^2\Big]\\
&= \|g\|_\infty^2 \sup_{n\geq 1} \mathbb{E}_{s,x}\Big[2\int_s^t \int_r^t \Big\{ \int_{\mathbb{R}^d} p_{\epsilon_n}(y,B_r)\mu_1(r,dy)\\
&\qquad\qquad \times \int_{\mathbb{R}^d} p_{\epsilon_n}(y,B_v)\mu_1(v,dy)\Big\}\mu_2(dv)\mu_2(dr)\Big]\\
&\leq \|g\|_\infty^2 \sup_{n\geq 1} \mathbb{E}_{s,x}\Big[2\int_s^t \Big\{ \int_{\mathbb{R}^d} p_{\epsilon_n}(y,B_r)\mu_1(r,dy)\\
&\qquad\qquad \times \int_r^t \int_{\mathbb{R}^d} p_{v-r+\epsilon_n}(y,B_r)\mu(dvdy)\Big\}\mu_2(dr)\Big]\\
&\leq \|g\|_\infty^2 \sup_{n\geq 1} \mathbb{E}_{s,x}\Big[2\int_s^t \Big\{ \int_{\mathbb{R}^d} p_{\epsilon_n}(y,B_r)\mu_1(r,dy)\; c_t\Big\}\mu_2(dr)\Big] \leq c_{t,g} < \infty.
\end{aligned}$$

That is, $(\int_s^t g(r,B_r)dA^{\epsilon_n}(r) : n \geq 1)$ is $L^2(\mathbb{P}_{s,x})$-bounded and so, by Lemma 3.5,

$$\Big(\int_s^t g(r,B_r)dA^{\epsilon_n}(r) : n \geq 1\Big) \text{ is uniformly integrable w.r.t. } \mathbb{P}_{s,x}. \tag{8.25}$$

Also, Fatou's lemma and (8.24) yield

$$\begin{aligned}
\mathbb{E}_{s,x}\Big[\int_s^t g(r,B_r)dA(r)\Big] &= \mathbb{E}_{s,x}\Big[\liminf_{n\to\infty}\int_s^t g(r,B_r)dA^{\epsilon_n}(r)\Big] \\
&\le \liminf_{n\to\infty}\mathbb{E}_{s,x}\Big[\int_s^t g(r,B_r)dA^{\epsilon_n}(r)\Big] \\
&\le \sup_{n\ge 1}\mathbb{E}_{s,x}\Big[\int_s^t g(r,B_r)dA^{\epsilon_n}(r)\Big] \;<\; \infty,
\end{aligned}$$

that is, $\int_s^t g(r,B_r)dA(r) \in L^1(\mathbb{P}_{s,x})$. So (8.23) and (8.25) imply $L^1(\mathbb{P}_{s,x})$-convergence of $\int_s^t g(r,B_r)dA^{\epsilon_n}(r)$ to $\int_s^t g(r,B_r)dA(r)$ (cf. Proposition 3.12 of [Kal97]). In particular,

$$\begin{aligned}
&\mathbb{E}_{s,x}\Big[\int_s^t g(r,B_r)dA(r)\Big] \\
&= \lim_{n\to\infty}\mathbb{E}_{s,x}\Big[\int_s^t g(r,B_r)dA^{\epsilon_n}(r)\Big] \\
&= \lim_{n\to\infty}\mathbb{E}_{s,x}\Big[\int_s^t\int_{\mathbb{R}^d} g(r,B_r)p_{\epsilon_n}(y,B_r)\mu(drdy)\Big] \\
&= \lim_{n\to\infty}\int_s^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} p_{r-s}(x,z)g(r,z)p_{\epsilon_n}(y,z)dz\;\mu(drdy)
\end{aligned}$$

It remains to show that the latter limit equals the r.h.s. of (8.11). For any $\gamma > 0$ we have

$$\begin{aligned}
&\Big|\int_s^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} p_{r-s}(x,z)g(r,z)p_{\epsilon_n}(y,z)dz\;\mu(drdy) - \int_s^t\int_{\mathbb{R}^d} p_{r-s}(x,y)g(r,y)\mu(drdy)\Big| \\
&\le\; \|g\|_\infty\int_s^{(s+\gamma)\wedge t}\int_{\mathbb{R}^d}\Big(p_{r-s+\epsilon_n}(x,y)+p_{r-s}(x,y)\Big)\mu(drdy) \\
&\quad+\int_{(s+\gamma)\wedge t}^t\int_{\mathbb{R}^d}\Big|P_{\epsilon_n}[p_{r-s}(x,.)g(r,.)](y)-p_{r-s}(x,y)g(r,y)\Big|\mu(drdy) \\
&=:\; I_1(\epsilon_n)+I_2(\epsilon_n).
\end{aligned}$$

Recall that $\mu(dtdx)$ satisfies condition (B). So, using Lemmas 4.2(i) $\Rightarrow$ (ii) and 4.4(i), $I_1(\epsilon_n)$ can easily be estimated by $\|g\|_\infty c_t\gamma^\beta$ (uniformly in n) where $\beta := \beta_1/2+\beta_2-d/2$. On the other hand, using the continuity of the semigroup (P_t) and the dominated convergence theorem, we can show that $I_2(\epsilon_n)$ tends to 0 as $n\to\infty$. Hence, for every $\bar\gamma > 0$ we can find some $n_{\bar\gamma}\ge 1$ such that $I_1(\epsilon_n)+I_2(\epsilon_n)\le\bar\gamma$ for all $n\ge n_{\bar\gamma}$. That is, $\lim_{n\to\infty}(I_1(\epsilon_n)+I_2(\epsilon_n))=0$. This completes the proof of Lemma 8.5. □

8.6 Some remarks

Let us mention an alternative construction of the CCP $C_{[B,\mu]}$ of a Brownian motion B and a measure $\mu(dtdx)$ satisfying condition (B). Set $h_{s,t}(x) := \int_s^t\int_{\mathbb{R}^d} p_{r-s}(x,y)\mu(drdy)$ for all

$0 \leq s \leq t$ and $x \in \mathbb{R}^d$. By means of Lemmas 4.2 - 4.8, Lemma 3.5 and Proposition 3.6 one can verify that h satisfies conditions $(i) - (iv)$ of Theorem 3.46. Hence, the theorem shows the existence of a CAF with characteristic h, i.e. the existence of $C_{[B,\mu]}$.

In the case $\mu(dtdx) = \mu_1(dx)dt$, where $\mu_1(dx) \in \mathcal{M}(\mathbb{R}^d)$ satisfies

$$\exists \alpha_1 \in (d-2, d] : \qquad \sup_{x \in \mathbb{R}^d} \int_{B[x,1]} |x-y|^{-\alpha_1} \mu_1(dx) < \infty, \tag{8.26}$$

Delmas ([Del96], Section 2) constructed a CAF A of B in the sense of (3.23) with moment

$$\mathbb{E}_x\Big[\int_0^t f(r,y)dA(r)\Big] = \int_0^t \int_{\mathbb{R}^d} p_r(x,y) f(r,y) \mu_1(dy)dr \tag{8.27}$$

for every $f \in C_b^+([0,\infty) \times \mathbb{R}^d)$ and $t \geq 0$. The construction is based on a general existence result for CAFs by Volkonsky ([Vol60]) which is very similar to Theorem 3.46. The CAF A is of course also a CAF in the sense of Definition 3.44. Setting $f := \mathbf{1}$ in (8.27) we see that A is nothing but the CCP of B and $\mu(dtdx) = \mu_1(dx)dt$. Note that condition (8.26) is equivalent to condition (B) for measures $\mu(dtdx) = \mu_1(dx)dt$ (recall Remark 2.9). Hence, Theorem 8.2 generalizes Delmas' result. Theorem 8.2 also provides an approximation of the CCP by the approximate CCP which can not be found in [Del96]. However, a similar approximation was shown early by Evans and Perkins ([EP94], Theorem 4.1) for measures $\mu(dtdx) = \mu_1(t,dx)dt$ satisfying

$$\lim_{\epsilon \downarrow 0} \sup_{t \geq 0} \sup_{x \in \mathbb{R}^d} \int_0^\epsilon \int_{\mathbb{R}^d} p_r(x,y) \mu_1(t+r, dy)dr = 0. \tag{8.28}$$

With help of Lemma 4.2 it is easy to show that every measure $\mu(dtdx) = \mu_1(t,dx)dt$, that satisfies condition (B), satisfies condition (8.28) as well. In fact, the class of measures $\mu(dtdx) = \mu_1(t,dx)dt$ satisfying condition (8.28) consists essentially of those measures $\mu(dtdx)$ of the form $\mu_1(t,dx)dt$ that satisfy condition (B). That means, Theorem 8.2 also generalizes Evans and Perkins' result, at least partially. In both references [Del96] and [EP94] the CCP is considered as a CAF in the sense of (3.23). This is possible since in these references $\mu_2(dt)$ is assumed to be the uniform measure dt. However, already when $\mu_1(t,dx)$ varies in t, i.e. when $\mu_1(.,dx) \not\equiv \mu_1(dx)$, a trick is needed. Evans and Perkins had to consider

$$C_{[B,\mu]}(t) \text{ “$=$” } \int_0^t \int_{\mathbb{R}^d} \delta_y(B_r)\mu_1(r,dy)dr = \int_0^t \Big(\int_{\mathbb{R}^d} \delta_y(B_r)\mu_1(r,dy)\Big)dr$$

as a CAF of the Markov process $((r,B_r) : r \geq 0)$ (instead of $(B_r : r \geq 0)$) in order to ensure property (3.23). In our general setting, where $\mu_2(dr)$ may differ from dr, this trick does not apply any more. Even when considering the process $((r,B_r) : r \geq 0)$, the inhomogeneity of $\mu_2(dt)$ causes a violation of (3.23). For this reason we had to adopt Dynkin's more general notion of CAFs.

9 Catalytic super-Brownian motion

In this chapter we study the so-called *catalytic super-Brownian motion* which arises as high-density/short-lifetime limit of critical binary branching Brownian particles whose branching times depend on a medium. We abbreviate super-Brownian motion by *SBM*. For an overview and historical notes on (catalytic) SBM we refer to [Daw93], [Dyn94], [LG99], [DF00], [Kle00b], [Eth00] or [Per02], see also references therein. A main issue of this chapter is to study the following question: For which catalysts does exist a jointly continuous (Lebesgue) density field for the corresponding catalytic SBM? We will go into detail w.r.t. to that question in Section 9.8. In particular we characterize the density field as unique solution to a certain SPDE (Section 9.9) and we specify the Laplace transforms and the first two moments of the density field (Sections 9.10, 9.11). Previously we shall give a direct construction of the catalytic SBM (with more general catalysts than considered in literature so far) and we describe the mentioned particle approximation in more detail (Sections 9.2, 9.3). In Sections 9.4, 9.5, 9.6 and 9.7 we obtain some features of the catalytic SBM: moments of any order, sample continuity, strong Markov property, collision measure with catalyst, martingale problem. In particular, we prove uniqueness of solutions to the martingale problem; this was an open problem for all singular catalysts. The mentioned direct construction relies on the fact that the catalytic SBM is a measure-valued branching process. For this reason we first discuss the notion of branching processes.

9.1 Measure-valued branching processes

The notion of branching processes with more general state space than $E = \mathbb{N}$ was introduced by Jiřina in 1958 ([Jiř58]). He considered the case $E = \mathbb{R}$; related works are [Lam67] and [Sil68]. In fact, the very first continuous state branching process was studied by Feller in 1951 ([Fel51]). He constructed the limit of a rescaled classical Galton-Watson process but he did not introduce the concept of continuous state branching. Branching processes with the yet more general state space $E = \mathcal{M}_f(S)$ were first considered by Watanabe ([Wat68]), Ikeda, Nagasawa and Watanabe ([INW69]) and Dawson ([Daw75]) under various assumptions on S. Today there exists a general theory of $\mathcal{M}_f(S)$-valued branching processes for a Polish (or Luzin) space S. Basic works have been provided by Dawson ([Daw92], [Daw93]), Dynkin ([Dyn91], [Dyn93], [Dyn94]), Dynkin, Kuznetsov and Skorohod ([DKS94]), Schied ([Sch99]) and others. Following these references we briefly study the notion of $\mathcal{M}_f(S)$-valued branching processes. For simplicity we assume $S = \mathbb{R}^d$.

Let $\bar{X} = [\bar{X}, \mathbb{P}_{s,\eta} : s \geq 0, \eta \in \mathcal{M}_f(\mathbb{R}^d)]$ be an $\mathcal{M}_f(\mathbb{R}^d)$-valued canonical Markov process (i.e. $[\Omega, \mathcal{F}] = [E^{[0,\infty)}, \mathcal{E}^{[0,\infty)}]$ with $E = \mathcal{M}_f(\mathbb{R}^d)$, and $\bar{X}$ is the coordinate process). Roughly speaking, $\bar{X}$ is called a *branching process* if the sum of two independent versions $\bar{X}^1$ and $\bar{X}^2$ of $\bar{X}$ with initial states η_1, respectively η_2, has the same law as $\bar{X}$ starting at $\eta = \eta_1 + \eta_2$. That is, we require for every $0 \leq s \leq t$ and $\eta_1, \eta_2 \in \mathcal{M}_f(S)$:

$$\text{law}(\bar{X}_t | \bar{X}_s = \eta_1 + \eta_2) = \text{law}(\bar{X}_t^1 + \bar{X}_t^2 | \bar{X}_s^1 = \eta_1, \bar{X}_s^2 = \eta_2). \tag{9.1}$$

Property (9.1) extends in the obvious way to initial states of the form $\eta = \eta_1 + \cdots + \eta_n$.

In particular, each state $\bar{X}_t$ of a branching process $\bar{X}$ is infinitely divisible[26] (choose $\eta_1 = \cdots = \eta_n = \eta/n$). As before we denote the corresponding Markov transition function by $\mu = \{\mu_{s,t} : 0 \le s \le t\}$; hence $\mathbb{P}_{s,\eta}[\bar{X}_t \in d\nu] = \mu_{s,t}(\eta, d\nu)$ for $0 \le s \le t$ and $\eta \in \mathcal{M}_f(\mathbb{R}^d)$. The *log-Laplace transform* $V^{\eta}_{s,t}$ of the transition probability $\mu_{s,t}(\eta, d\nu)$ is defined by

$$\psi \mapsto V^{\eta}_{s,t}(\psi) := -\log \mathbb{E}_{s,\eta}\Big[e^{-\langle \bar{X}_t, \psi\rangle}\Big], \qquad C_b^+(\mathbb{R}^d) \to \mathbb{R}_+.$$

Taking Proposition 3.31 into account, it is easy to show that (9.1) holds if and only if

$$V^{\eta_1+\eta_2}_{s,t}(\psi) = V^{\eta_1}_{s,t}(\psi) + V^{\eta_2}_{s,t}(\psi) \qquad \forall \psi \in C_b^+(\mathbb{R}^d). \tag{9.2}$$

Relation (9.2) is called the *(weak) branching property.* From an analytical point of view the situation gets easier when restricting to branching processes which satisfy

$$x \mapsto V^{\delta_x}_{s,t}(\psi) \in C_b^+(\mathbb{R}^d) \qquad \forall \psi \in C_b^+(\mathbb{R}^d) \tag{9.3}$$

and

$$V^{\eta}_{s,t}(\psi) = \int_{\mathbb{R}^d} V^{\delta_x}_{s,t}(\psi)\eta(dx) \qquad \forall \psi \in C_b^+(\mathbb{R}^d) \tag{9.4}$$

for every $0 \le s \le t$ and $\eta \in \mathcal{M}_f(\mathbb{R}^d)$. Condition (9.4) is called *strong branching property* and clearly implies (9.2). The Markov process $\bar{X}$ is called *strong branching process* if the log-Laplace transforms of its transition probabilities satisfy (9.3) and (9.4). If $\bar{X}$ is a strong branching process, we use the notation $U_{s,t}\psi(x) := V^{\delta_x}_{s,t}(\psi)$. In view of the following lemma, $(U_{s,t})_{0 \le s \le t}$ is called the *log-Laplace semigroup* associated with $\bar{X}$.

Lemma 9.1 *Let $\bar{X}$ be a strong branching process. Then $(U_{s,t})_{0\le s\le t}$ provides a semigroup on $C_b^+(\mathbb{R}^d)$. That is, $U_{t,t} = \mathbb{I}$ and $U_{s,v}U_{v,t} = U_{s,t}$ hold on $C_b^+(\mathbb{R}^d)$ for all $0 \le s \le v \le t$.*

The proof is a simple application of $\bar{X}$'s Markov property. We omit it here. By the definitions of the log-Laplace transform and the log-Laplace semigroup, $\mu_{s,t}(\eta, d\nu)$'s Laplace transform can be written as

$$L_{\mu_{s,t}(\eta,d\nu)}(\psi)\ \Big(= \mathbb{E}_{s,\eta}\Big[e^{-\langle \bar{X}_t,\psi\rangle}\Big]\Big)\ = e^{-V^{\eta}_{s,t}(\psi)} = e^{-\langle \eta, U_{s,t}\psi\rangle}.$$

Hence, the log-Laplace semigroup $(U_{s,t})_{0\le s\le t}$ determines the law of $\bar{X}$; recall Remark 3.48. On the other hand, we know from Proposition 3.50 that any inhomogeneous semigroup $(U_{s,t}) \equiv (U_{s,t})_{0\le s\le t}$ on $C_b^+(\mathbb{R}^d)$ which satisfies

$$\psi \mapsto U_{s,t}\psi(x) \text{ is negative definite,} \qquad \forall 0 \le s \le t \text{ and } x \in \mathbb{R}^d \tag{9.5}$$

and

$$\psi \mapsto U_{s,t}\psi \text{ is } bp\text{-continuous,} \qquad \forall 0 \le s \le t \tag{9.6}$$

induces an $\mathcal{M}_f(\mathbb{R}^d)$-valued Markov process whose transition probability $\mu_{s,t}(\eta, d\nu)$ has Laplace transform $L_{\mu_{s,t}(\eta,d\nu)}(\psi) = e^{-\langle \eta, U_{s,t}\psi\rangle}$. By the form of the Laplace transform, this process is easily seen to be a strong branching process. That means any inhomogeneous semigroup $(U_{s,t})$ on $C_b^+(\mathbb{R}^d)$, which satisfies (9.5) and (9.6), is the log-Laplace semigroup associated with some strong branching process.

[26] A finite random measure ξ is said to be *infinitely divisible* if for every integer n there are independent identically distributed finite random measures $\xi_1, \ldots, \xi_n$ such that $\xi \overset{\mathrm{d}}{=} \xi_1 + \cdots + \xi_n$, (cf. [Daw92], p.29).

9.2 Direct construction and some comments

In this section we give a rigorous definition and a direct construction (i.e. without referring to any approximating particle system) of the catalytic SBM. Pick a measure $\varrho(dtdx) \in \mathcal{M}([0,\infty)\times\mathbb{R}^d)$ satisfying condition (B) from Definition 2.22. We call $\varrho(dtdx)$ (admissible) *catalyst*. This name will be justified in the next section. For $\psi \in C_b^+(\mathbb{R}^d)$ denote the unique $C_b^+(\mathbb{R}^d)$-valued solution (in the sense of Definition 7.8) to the following BPDE

$$\begin{aligned} -\frac{\partial}{\partial s}u(s,t,x) &= \frac{1}{2}\Delta u(s,t,x) - \frac{1}{2}u^2(s,t,x)\frac{\varrho(dsdx)}{dsdx}(s,x) \\ u(t,t,x) &= \psi(x) \qquad s\in[0,t],\ x\in\mathbb{R}^d \end{aligned} \tag{9.7}$$

by $(U_{s,t}\psi(x) : s \in [0,t], x \in \mathbb{R}^d)$. The existence and uniqueness is guaranteed by Theorem 7.7 and Remark 7.9. In view of (7.16), we can regard BPDE (9.7) as

$$u(s,t,x) = P_{t-s}\psi(x) - \frac{1}{2}\int_s^t \int_{\mathbb{R}^d} p_{r-s}(x,y)u^2(r,t,y)\varrho(drdy), \quad s\in[0,t], x\in\mathbb{R}^d. \tag{9.8}$$

From Lemma 7.10 we know that $(U_{s,t})$ provides an inhomogeneous semigroup of operators on $C_b^+(\mathbb{R}^d)$. At the end of this section we will also prove:

Lemma 9.2 *The map $\psi \mapsto U_{s,t}\psi$ is bp-continuous for every $0 \le s \le t$. Moreover, the map $\psi \mapsto U_{s,t}\psi(x)$ is negative definite for every $0 \le s \le t$ and $x \in \mathbb{R}^d$.*

Hence, $(U_{s,t})$ is an inhomogeneous semigroup on $C_b^+(\mathbb{R}^d)$ which satisfies (9.5) and (9.6). Thus, as seen in the previous section, $(U_{s,t})$ is the log-Laplace semigroup associated with some $\mathcal{M}_f(\mathbb{R}^d)$-valued strong branching process $\bar{X}$. So we can define:

Definition 9.3 [CATALYTIC SBM] *Let $\varrho(dtdx) \in \mathcal{M}([0,\infty)\times\mathbb{R}^d)$ satisfy condition* (B) *and denote the unique $C_b^+(\mathbb{R}^d)$-valued solution to BPDE (9.7) (i.e., more precisely, to (9.8)) by $(U_{s,t}\psi(x) : s\in[0,t], x\in\mathbb{R}^d)$. The (canonical) $\mathcal{M}_f(\mathbb{R}^d)$-valued strong branching process $\bar{X} = [\bar{X}, \mathbb{P}_{s,\eta} : 0 \le s \le t, \eta \in \mathcal{M}_f(\mathbb{R}^d)]$ with associated log-Laplace semigroup $(U_{s,t})$ is called catalytic SBM with catalyst $\varrho(dtdx)$.*

Hence, the catalytic SBM $\bar{X}$ with catalyst $\varrho(dtdx)$ is the $\mathcal{M}_f(\mathbb{R}^d)$-valued Markov process whose transition probability $\mu_{s,t}(\eta, d\nu)$ $(0 \le s \le t,\ \eta \in \mathcal{M}_f(\mathbb{R}^d))$ has Laplace transform

$$L_{\mu_{s,t}(\eta,d\nu)}(\psi) \left(= \mathbb{E}_{s,\eta}\Big[e^{-\langle \bar{X}_t,\psi\rangle}\Big]\right) = e^{-\langle \eta, U_{s,t}\psi\rangle}, \qquad \psi\in C_b^+(\mathbb{R}^d) \tag{9.9}$$

where $(U_{s,t}\psi(x) : s\in[0,t], x\in\mathbb{R}^d)$ denotes the unique $C_b^+(\mathbb{R}^d)$-valued solution to (9.8). We also refer to $\bar{X}$ as *reactant*. A particle approximation will be presented in the next section. Before proving Lemma 9.2 we give some comments yet.

The classical SBM (i.e. $\varrho(dtdx) = dtdx$) first appeared in works of Watanabe ([Wat68]) and Dawson ([Daw75], [Daw77]). Therefore it is nowadays known as *Dawson-Watanabe process*. The classical SBM describes an infinitesimal system of independent critical binary branching Brownian particles where the branching intensity is homogeneous in space and time. At the end of the 1980's Dawson proposed to study a modified system where the branching is governed by a (possibly singular) measure $\varrho(dtdx)$, i.e. where the branching intensity at time t at location x is given by "$\frac{\varrho(dtdx)}{dtdx}(t,x)$". Dawson and Fleischmann ([DF91], [DF92], [DF97]) constructed the corresponding measure-valued process for measures (catalysts) of the form $\varrho(dtdx) = \varrho_1(t,dx)dt$ and called it catalytic SBM. The existence of the reactant is coupled with the existence of a "good" collision process of the Brownian motion and the catalyst $\varrho(dtdx)$ (the notion of collision processes was studied in Chapter 8). This will become apparent in the next section. In dimension $d = 1$ there is a large class of admissible catalysts. In fact, one can even work with a spatially atomic catalyst since the collision process of a Brownian motion and a spatial Dirac measure (i.e. the Brownian local time) exists. In higher dimensions ($d \geq 2$) one needs to be a bit more careful. For instance, it is well known that the Brownian local time in dimensions $d \geq 2$ does not exist. So a spatial Dirac measure $\delta_c(dx)dt$ cannot be used as a catalyst. However, in Section 8.3 we have seen that any measure $\varrho(dtdx)$, which satisfies condition (B), possesses a "good" continuous collision process with a Brownian motion; "good" means that it satisfies (8.12) (whereby it is a branching functional in the sense of [Dyn94], cf. the next section). This is the reason why we required the catalyst to satisfy condition (B). Related admissibility conditions have been given earlier by Evans and Perkins and Delmas, see (8.28), respectively (8.26). Dawson and Fleischmann ([DF97]) adapted condition (8.28). In Section 8.6 we compared these conditions with condition (B). We have seen that (B) is weaker than (8.26) and weaker, at least partially, than (8.28). In particular, a catalytic SBM whose catalyst $\varrho(dtdx) = \varrho_1(t,dx)\varrho_2(dt)$ is not Lebesgue in time (i.e. $\varrho_2(dt) \neq dt$) has not been constructed so far.

To complete our direct construction we have to prove Lemma 9.2 yet.

Proof (of Lemma 9.2) The solution $(U_{s,t}\psi(x) : s \in [0,t], x \in \mathbb{R}^d)$ of BPDE (9.7) was constructed by means of the Picard-Lindelöf iteration:

$$\begin{aligned} U_{s,t}^{(0)}\psi(x) &:= P_{t-s}\psi(x), \\ U_{s,t}^{(n)}\psi(x) &:= P_{t-s}\psi(x) - \frac{1}{2}\int_s^t \int_{\mathbb{R}^d} p_{r-s}(x,y)(U_{r,t}^{(n-1)}\psi)^2(y)\varrho(drdy), \qquad n \geq 1 \end{aligned}$$

(cf. Sections 7.2 and 7.6). For any $K > 0$, the solution $U_{.,t}\psi(.)$ can be approximated by $U_{.,t}^{(n)}\psi(.)$ uniformly in $\psi \in C_{b,K}^+(\mathbb{R}^d)$ where $C_{b,K}^+(\mathbb{R}^d) := \{\phi \in C_b^+(\mathbb{R}^d) : \|\phi\|_\infty \leq K\}$. Indeed, using Lemma 4.2$(i) \Rightarrow (ii)$ we get

$$\|U_{s,t}\psi(.) - U_{s,t}^{(n+1)}\psi(.)\|_\infty \leq c_{t,K}\int_s^t \frac{1}{(r-s)^{d/2-\beta_1/2}}\|U_{r,t}\psi(.) - U_{r,t}^{(n)}\psi(.)\|_\infty \varrho_2(dr)$$

for all $s \in [0,t]$, $\psi \in C_{b,K}^+(\mathbb{R}^d)$ and $n \geq 1$, for some constant $c_{t,K} > 0$. By means of the Gronwall-type Lemma 4.11 we deduce

$$\sup_{s\leq t}\|U_{s,t}\psi(.) - U_{s,t}^{(n)}\psi(.)\|_\infty \leq \tilde{c}_{t,K}\, q_{t,K}^n$$

for all $\psi \in C_{b,K}^+(\mathbb{R}^d)$ and $n \geq 1$, for some constants $\tilde{c}_{t,K} > 0$ and $q_{t,K} \in (0,1)$. That is,

$$\lim_{n\to\infty}\sup_{s\leq t}\|U_{s,t}\psi(.) - U_{s,t}^{(n)}\psi(.)\|_\infty = 0 \quad \text{uniformly in } \psi \in C_{b,K}^+(\mathbb{R}^d). \tag{9.10}$$

We are know in the position to prove the *bp*-continuity of $\psi \mapsto U_{s,t}\psi$. First of all note that $\psi \mapsto P_{t-s}\psi$ is *bp*-continuous for every $s \in [0,t]$. Using the recursive definition of the $U_{s,t}^{(n)}\psi$ and the dominated convergence theorem, one can easily deduce *bp*-continuity of $\psi \mapsto U_{s,t}^{(n)}\psi$ for every $n \geq 1$. Now, pick some $\psi \in C_b^+(\mathbb{R}^d)$ and any sequence (ψ_m) in $C_b^+(\mathbb{R}^d)$ which *bp*-converges to ψ. Note that there exists some constant $K = K_\psi > 0$ such that $\psi, \psi_1, \psi_2, \ldots \in C_{b,K}^+(\mathbb{R}^d)$. By (9.10) and the *bp*-continuity of $\phi \mapsto U_{s,t}^{(n)}\phi$ ($\forall n \geq 1$) we obtain for every $x \in \mathbb{R}^d$:

$$\begin{aligned} U_{s,t}\psi(x) &= \lim_{n\to\infty} U_{s,t}^{(n)}\psi(x) \\ &= \lim_{n\to\infty}\lim_{m\to\infty} U_{s,t}^{(n)}\psi_m(x) \;=\; \lim_{m\to\infty}\lim_{n\to\infty} U_{s,t}^{(n)}\psi_m(x) \;=\; \lim_{m\to\infty} U_{s,t}\psi_m(x). \end{aligned}$$

Hence $U_{s,t}\psi_m$ *bp*-converges to $U_{s,t}\psi$ as $m \to \infty$, and so $\psi \mapsto U_{s,t}\psi$ is *bp*-continuous.

It remains to prove the negative definiteness of $\psi \mapsto U_{s,t}\psi(x)$. We proceed in 3 steps.

Step 1. We first assume $\varrho(dtdx) = \varrho(x)dxdt$ for some $\varrho \in C_{b,+}^\infty(\mathbb{R}^d)$. In this case, BPDE (9.7) can be associated with the following PDE

$$\begin{aligned} \frac{\partial}{\partial t}u(t,x) &= \frac{1}{2}\Delta u(t,x) - \frac{1}{2}u^2(t,x)\varrho(x) \\ u(0,x) &= \psi(x) \qquad t \geq 0,\ x \in \mathbb{R}^d. \end{aligned} \tag{9.11}$$

Equation (9.11) is equivalent to the integral equation

$$u(t,x) = P_t\psi(x) - \frac{1}{2}\int_0^t \int_{\mathbb{R}^d} p_{t-r}(x,y)u^2(r,y)\varrho(y)dydr. \tag{9.12}$$

Let us consider an approximate version of (9.11), namely

$$\begin{aligned} \frac{\partial}{\partial t}u(t,x) &= \frac{1}{2}\Delta u(t,x) + \frac{1}{h^2}\Big[1 - e^{-hu(t,x)} - hu(t,x)\Big]\varrho(x) \\ u(0,x) &= \psi(x) \qquad t \geq 0,\ x \in \mathbb{R}^d \end{aligned} \tag{9.13}$$

($h > 0$). Equation (9.13) is clearly equivalent to the integral equation

$$u(t,x) = P_t\psi(x) + \int_0^t \int_{\mathbb{R}^d} p_{t-r}(x,y)\frac{1}{h^2}\Big[1 - e^{-hu(r,y)} - hu(r,y)\Big]\varrho(y)dydr. \tag{9.14}$$

But it is also equivalent to[27]

$$u(t,x) = e^{-\frac{\varrho(x)}{h}t}P_t\psi(x) + \int_0^t \int_{\mathbb{R}^d} e^{-\frac{\varrho(x)}{h}(t-r)}p_{t-r}(x,y)\Big[1 - e^{-hu(r,y)}\Big]\frac{\varrho(y)}{h^2}dydr. \tag{9.15}$$

The latter equation can be solved by means of the Picard-Lindelöf iteration:

$$\begin{aligned} U_t^{(0)}\psi(x) &:= e^{-\frac{\varrho(x)}{h}t}P_t\psi(x), \\ U_t^{(n)}\psi(x) &:= e^{-\frac{\varrho(x)}{h}t}P_t\psi(x) + \int_0^t \int_{\mathbb{R}^d} e^{-\frac{\varrho(x)}{h}(t-r)}p_{t-r}(x,y)\Big[1 - e^{-hU_r^{(n-1)}\psi(y)}\Big]\frac{\varrho(y)}{h^2}dydr \end{aligned}$$

[27] Note that (Q_t) is the semigroup generated by $L := \frac{1}{2}\Delta - \frac{\varrho}{h}\mathbb{I}$, where $Q_t\psi(x) := e^{-(\varrho(x)/h)t}P_t\psi(x)$.

$(n \geq 1)$. For every $t \geq 0$ and $x \in \mathbb{R}^d$, the map $\psi \mapsto U_t^{(0)}\psi(x)$ is negative definite by the additivity of $\psi \mapsto P_t\psi$. Therefore, we successively obtain[28] negative definiteness of $\psi \mapsto U_t^{(n)}\psi(x)$ for all $n \geq 1$. Hence, if $(U_t^h\psi(x) : t \geq 0, x \in \mathbb{R}^d)$ denotes the unique non-negative solution to (9.13) (resp. (9.15)), we get negative definiteness of $\psi \mapsto U_t^h\psi(x)$ for all $t \geq 0$ and $x \in \mathbb{R}^d$. Here we exploit the fact that negative definiteness is closed under the limit. For the same reason we get negative definiteness of $\psi \mapsto U_t\psi(x)$ where $(U_t\psi(x) : t \geq 0, x \in \mathbb{R}^d)$ denotes the unique non-negative solution to (9.11) (resp. (9.12)). Indeed, $U_t^h\psi(x)$ converges to $U_t\psi(x)$ as $h \downarrow 0$ since $\frac{1}{h^2}[1 - e^{-hu} - hu] \to -\frac{1}{2}u^2$ as $h \downarrow 0$ uniformly in $u \in [0, K]$ $(\forall K > 0)$. To show this precisely, use (9.12), (9.14) and the classical Gronwall lemma. We omit the details.

Step 2. We next assume $\varrho(dtdx) = \varrho(t,x)dxdt$ for some $\varrho \in C_{b,+}^\infty([0,\infty) \times \mathbb{R}^d)$. Let ϱ_m denote a temporally piecewise constant approximation[29] of ϱ and let $(U_{s,t}^m\psi(x) : s \in [0,t], x \in \mathbb{R}^d)$ denote the unique non-negative solution to

$$u(s,t,x) = P_{t-s}\psi(x) - \frac{1}{2}\int_s^t \int_{\mathbb{R}^d} p_{r-s}(x,y)u^2(r,t,y)\varrho_m(r,y)dydr. \tag{9.16}$$

Proceeding (piecewise) as in Step 1 and taking the semigroup property of $(U_{s,t}^m)$ into account (recall Lemma 7.10), we obtain negative definiteness of $\psi \mapsto U_{s,t}^m\psi(x)$ for all $0 \leq s \leq t$, $x \in \mathbb{R}^d$ and $m \geq 1$. Again using the classical Gronwall lemma one can easily show that $U_{s,t}^m\psi(x)$ converges to $U_{s,t}\psi(x)$ as $m \to \infty$ where $(U_{s,t}\psi(x) : s \in [0,t], x \in \mathbb{R}^d)$ denotes the unique non-negative solution to (9.16) with ϱ_m replaced by ϱ. Consequently, $\psi \mapsto U_{s,t}\psi(x)$ is negative definite for all $0 \leq s \leq t$ and $x \in \mathbb{R}^d$; recall that negative definiteness is closed under the limit.

Step 3. Finally, let us consider the general case, i.e. $\varrho(dtdx)$ only has to satisfy condition (B). Set $\varrho_\epsilon(t,x) := \int_{-\infty}^\infty \int_{\mathbb{R}^d} p_\epsilon(t,r)p_\epsilon(x,y)\varrho(drdy)$ and let $(U_{s,t}^\epsilon\psi(x) : s \in [0,t], x \in \mathbb{R}^d)$ denote the unique non-negative solution to (9.8) with $\varrho(drdy)$ replaced by $\varrho_\epsilon(r,y)dydr$. From Step 2 we know that $\psi \mapsto U_{s,t}^\epsilon\psi(x)$ is negative definite for all $0 \leq s \leq t$ and $x \in \mathbb{R}^d$. Also, in Lemma 9.24 below we shall show $U_{s,t}^\epsilon\psi(x) \to U_{s,t}\psi(x)$ as $\epsilon \downarrow 0$ where $(U_{s,t}\psi(x) : s \in [0,t], x \in \mathbb{R}^d)$ denotes the unique non-negative solution to (9.8). Since negative definiteness is closed under the limit, we get negative definiteness of $\psi \mapsto U_{s,t}\psi(x)$ for all $0 \leq s \leq t$ and $x \in \mathbb{R}^d$. This completes the proof of Lemma 9.2. □

9.3 Particle approximation (Medium-dependent branching)

In this section we focus on an alternative construction of the catalytic SBM. The latter can be defined as the high-density/short-lifetime limit of a certain branching Brownian particle system. Since we already gave a rigorous construction in the previous section, we feel free to work only on an informal level. Our main goal is to give an intuition.

[28] Recall Lemma 3.49 and note that $\psi \mapsto \int L_{(a)}\psi\mu(da)$ is negative definite when $\psi \mapsto L_{(a)}\psi$ is negative definite for every a. Also, the sum of two negative definite functionals is again negative definite.

[29] For instance, set $\varrho_m(t,x) := \varrho(\frac{k}{m}, x)$ where $k \in \{0,1,2,\ldots\}$ is chosen in such a way that $t \in [\frac{k}{m}, \frac{k+1}{m})$.

We shall describe an approximating particle system for a more general measure-valued branching process than the catalytic SBM, namely for the $(B, A, (.)^2)$-superprocess. Consider a system of particles moving according to Brownian motion B through $\mathbb{R}^d$, producing either 0 or 2 offspring – both with the same chance (*critical binary branching*) – at their death times and carrying unit mass 1 each. The death time and the death site of a parent coincide with the birth time and the birth site of its immediate descendants. Apart from the birth times and the birth sites, the descendants are independent copies of the parent, i.e. all particles undergo the same random mechanism. The lifetime of a particle is governed as follows. Each new born particle independently gets assigned a lifetime ζ by an exponential distribution with parameter 1. Its individual "branching age" is given by a CAF A of Brownian motion[30] with characteristic h satisfying[31]

$$h_{s,t}(x) < \infty \quad \forall s,t,x \qquad \text{and} \qquad \lim_{s,t\to r} h_{s,t}(x) = 0 \ \ (\text{uniformly in } x) \ \ \forall r \tag{9.17}$$

(following Dynkin we say a CAF A is a *branching functional* if it satisfies (9.17)). That is, a particle born at time s lives until $(A(s,t] : t \geq s)$ first exceeds ζ, i.e. the actual lifetime is $\zeta_A := \inf\{t > s : A(s,t] > \zeta\} - s$. In other words, if a particle is alive at time s, then the conditional probability for the event that the particle survives the interval $(s,t]$ and dies in $(t, t+h]$, conditioned on its trajectory $(B_r : r \in [s, t+h])$, equals[32]

$$\begin{aligned} & e^{-A(s,t+h]} - e^{-A(s,t]} \\ = \; & e^{-A(s,t]}\{e^{-(A(s,t+h]-A(s,t])} - 1\} \\ = \; & e^{-A(s,t]}\left\{A(t,t+h] \; + \; o(A(t,t+h])\right\}. \end{aligned}$$

In particular, the conditional probability for the event that a particle survives the interval $(s,t]$ and dies in $(t, t+dt]$ equals $e^{-A(s,t]}dA(t)$. Of course, this is only a more or less heuristic description of the system. More rigorously, we think of the system as an $\mathcal{N}_f(\mathbb{R}^d)$-valued process $\bar{X}^1$ where $\mathcal{N}_f(\mathbb{R}^d)$ denotes the subspace of $\mathcal{M}_f(\mathbb{R}^d)$ consisting of all finite sums of Dirac measures. If there are k initial particles located at sites $x_1, \ldots, x_k \in \mathbb{R}^d$, then we identify the initial state with the measure $\sum_{i=1}^k \delta_{x_i}(dx)$. More generally, the state of the process at time t is wanted to be

$$\bar{X}^1_t(dx) = \sum_{i \sim t} \delta_{B_i(t)}$$

where the sum ranges over all particles that are alive at time t and $B_i(t)$ denotes the location of the particle with index i at time t. The process $\bar{X}^1$ in mind should certainly be a Markov process since Brownian motion and the exponential lifetime are Markovian. One can justify[33] that $\bar{X}^1$ can be identified with the $\mathcal{N}_f(\mathbb{R}^d)$-valued Markov process $[\bar{X}^1, \mathbb{P}^1_{s,\eta_1} :$

[30] To approximate the catalytic SBM, we shall choose A to be the CCP of Brownian motion and $\varrho(dtdx)$.

[31] Recently Klenke ([Kle02]) weakened condition (9.17). In particular, A can be chosen to be a discontinuous additive functional of Brownian motion.

[32] Note that $s + \zeta_A \in (t, t+h]$ if and only if $A(s,t] < \zeta \leq A(s, t+h]$. Also, $o(A(t,t+h])$ is defined to satisfy $\lim_{h\downarrow 0} o(A(t,t+h])/A(t,t+h] = 0$; note that $\lim_{h\downarrow 0} A(t,t+h] = 0$ by the continuity of A.

[33] See e.g. Chapter 4 of [Daw93], Chapter 3 of [Dyn94], [Del96] or [Kle02].

$0 \le s \le t, \eta_1 \in \mathcal{N}_f(\mathbb{R}^d)]$ which is cadlag and whose transition probability $\mu_{s,t}(\eta_1, d\nu)$ $(0 \le s \le t,\ \eta_1 \in \mathcal{N}_f(\mathbb{R}^d))$ has Laplace transform

$$L_{\mu_{s,t}(\eta_1,d\nu)}(\psi)\ \Big(= \mathbb{E}^1_{s,\eta_1}\Big[e^{-\langle \bar{X}^1_t, \psi\rangle}\Big]\Big) \ = e^{\langle \eta_1, \log W^1_{s,t}\psi\rangle}, \qquad \psi \in C_b^+(\mathbb{R}^d)$$

where $(W^1_{s,t}\psi(x) : s \in [0,t], x \in \mathbb{R}^d)$ is the unique non-negative solution to

$$w(s,t,x) = P_{t-s}e^{-\psi}(x) + \frac{1}{2}\pi_{s,x}\Big[\int_s^t (1 - w(r,t,B_r))^2 dA(r)\Big], \quad s \in [0,t], x \in \mathbb{R}^d. \quad (9.18)$$

(Here $\pi_{s,x}$ denotes the law of Brownian motion starting at time s at site $x \in \mathbb{R}^d$.)

For the sake of a diffusion limit we modify the system as follows. For (fixed) $n \in \mathbb{N}$, let any particle carry unit mass $\frac{1}{n}$ instead of 1 and let the lifetime be assigned by an exponential distribution with parameter n instead of 1. We denote the corresponding Markov process by $\bar{X}^n$ and write $\mathbb{P}^n_{s,\eta_n}$ for the law of $\bar{X}^n$ starting at time s at $\eta_n \in \frac{1}{n}\mathcal{N}_f(\mathbb{R}^d)$. That is, $\bar{X}^n$ is the $\frac{1}{n}\mathcal{N}_f(\mathbb{R}^d)$-valued cadlag Markov process with the following Laplace transform of its transition probability $\mu_{s,t}(\eta_n, d\nu)$ $(0 \le s \le t,\ \eta_n \in \frac{1}{n}\mathcal{N}_f(\mathbb{R}^d))$:

$$L_{\mu_{s,t}(\eta_n,d\nu)}(\psi)\ \Big(= \mathbb{E}^n_{s,\eta_n}\Big[e^{-\langle \bar{X}^n_t, \psi\rangle}\Big]\Big) \ = e^{\langle n\eta_n, \log W^n_{s,t}\psi\rangle}, \qquad \psi \in C_b^+(\mathbb{R}^d)$$

where $(W^n_{s,t}\psi(x) : s \in [0,t], x \in \mathbb{R}^d)$ is the unique non-negative solution to equation (9.18) with ψ and A replaced by $\frac{1}{n}\psi$, respectively nA. Now assume the initial particles of the process $(\bar{X}^n_t : t \ge s)$ are distributed by a Poisson random measure with intensity measure $n\eta$ for some $\eta \in \mathcal{M}_f(\mathbb{R}^d)$. If $\mathbb{P}^n_{s,Poiss(n\eta)/n}$ denotes the law of the latter Markov process, then one can show that the sequence $(\mathbb{P}^n_{s,Poiss(n\eta)/n})$ converges weakly in $D([0,\infty), \mathcal{M}_f(\mathbb{R}^d))$, as $n \to \infty$, to some law $\mathbb{P}_{s,\eta}$, where $\bar{X} = [\bar{X}, \mathbb{P}_{s,\eta} : 0 \le s \le t, \eta \in \mathcal{M}_f(\mathbb{R}^d)]$ forms an $\mathcal{M}_f(\mathbb{R}^d)$-valued cadlag Markov process; recall that $\mathcal{M}_f(\mathbb{R}^d)$ is equipped with the weak topology. For the convergence in the cadlag space see Theorem 4.6.2 of [Daw93] or Theorem 4 of [Sch99]. An approximation of $\bar{X}$ in a weaker sense (convergence of the finite-dimensional distributions) can be found in Section 4.4 of [Daw93] or Section 3.3 of [Dyn94]. The transition probability $\mu_{s,t}(\eta, d\nu)$ $(0 \le s \le t,\ \eta \in \mathcal{M}_f(\mathbb{R}^d))$ of $\bar{X}$ has Laplace transform

$$L_{\mu_{s,t}(\eta,d\nu)}(\psi)\ \Big(= \mathbb{E}_{s,\eta}\Big[e^{-\langle \bar{X}_t, \psi\rangle}\Big]\Big) \ = e^{-\langle \eta, U_{s,t}\psi\rangle}, \quad \psi \in C_b^+(\mathbb{R}^d) \qquad (9.19)$$

where $(U_{s,t}\psi(x) : 0 \le s \le t, x \in \mathbb{R}^d)$ is the unique non-negative solution to

$$u(s,t,x) = P_{t-s}\psi(x) - \frac{1}{2}\pi_{s,x}\Big[\int_s^t u^2(r,t,B_r) dA(r)\Big], \quad s \in [0,t], x \in \mathbb{R}^d. \qquad (9.20)$$

In [Daw93] and [Dyn94] the limit process $\bar{X}$ is called $(B, A, (.)^2)$-*superprocess.*

Let us now consider a special branching functional A. Let $\varrho(dtdx) \in \mathcal{M}([0,\infty) \times \mathbb{R}^d)$ satisfy condition (B) (cf. Definition 2.22). In that case the collision process $C_{[B,\varrho]}$ of a Brownian motion B and $\varrho(dtdx)$ exists by Theorem 8.2. In particular, $C_{[B,\varrho]}$ is a CAF satisfying (9.17) (cf. (8.12)), i.e. a branching functional. Since $U_{.,t}\psi(.)$ is non-negative, it is also bounded. Hence, by moment formula (8.11) we can write equation (9.20) as

$$u(s,t,x) = P_{t-s}\psi(x) - \frac{1}{2}\int_s^t \int_{\mathbb{R}^d} p_{t-r}(x,y)u^2(r,t,y)\varrho(drdy), \quad s \in [0,t], x \in \mathbb{R}^d.$$

Accordingly, the $(B, C_{[B,\varrho]}, (.)^2)$-superprocess is nothing but the catalytic SBM with catalyst $\varrho(dtdx)$. In particular, the catalytic SBM is the diffusion limit of the above particle system where the particles "age" according to the CCP $C_{[B,\varrho]}$. That means the branching time of a particle strongly depends on the collision with $\varrho(dtdx)$. This is why the measure $\varrho(dtdx)$ is called catalyst. The described particle approximation yields in particular that the catalytic SBM has cadlag samples. From the direct construction in Section 9.2 we cannot immediately deduce this statement. However, in Section 9.5 we will show that the catalytic SBM is not only cadlag but may even be assumed to be continuous w.r.t. the weak topology. The key tool will be given in the next section (moment formula (9.28)).

9.4 Moments

We here focus on the moments of the catalytic SBM. Many of the arguments in this section have already been used by Delmas ([Del96]) for the case $\varrho(dtdx) = \varrho_1(dx)dt$. Let $\varrho(dtdx) \in \mathcal{M}([0,\infty) \times \mathbb{R}^d)$ be an admissible catalyst, i.e. satisfy condition (B), and $\bar{X}$ be the corresponding catalytic SBM. First we establish a moment formula for $\langle \bar{X}_t, \psi \rangle$ (Theorem 9.5). Afterwards we will see that this formula can be generalized to moments of sums $\sum_{i=1}^l \langle \bar{X}_{t_i}, \psi_i \rangle$ (Theorem 9.6) and even of integrals $\int_s^v \langle \bar{X}_r, f(r,.)\rangle dr$ (Corollary 9.8). The proofs of all these results rely on the following lemma.

Lemma 9.4 *Fix $t \geq 0$ and $J_t \in B_b([0,t]\times\mathbb{R}^d)$. Define the mappings $(s,x) \mapsto a_n(s,x|t,J_t)$, $n \geq 1$, recursively as follows (for $s \in [0,t]$, $x \in \mathbb{R}^d$):*

$$\begin{aligned} a_1(s,x|t,J_t) &:= J_t(s,x), \\ a_n(s,x|t,J_t) &:= -\frac{1}{2}\int_s^t \int_{\mathbb{R}^d} p_{r-s}(x,y)\Big(\sum_{j=1}^{n-1} a_j(r,y|t,J_t)a_{n-j}(r,y|t,J_t)\Big)\varrho(drdy), \quad n \geq 2. \end{aligned}$$

Then there exists some $\theta_t = \theta_{t,J_t} > 0$ such that the power series

$$u(s,x|t,J_t,\theta) := \sum_{n=1}^{\infty} a_n(s,x|t,J_t)\theta^n$$

converges absolutely for all $\theta \in [0,\theta_t)$ uniformly in $s \in [0,t]$ and $x \in \mathbb{R}^d$. Moreover, for every $\theta \in [0,\theta_t)$, $(u(s,x|t,J_t,\theta) : s \in [0,t], x \in \mathbb{R}^d)$ is the unique bounded solution of

$$u(s,x) = \theta J_t(s,x) - \frac{1}{2}\int_s^t \int_{\mathbb{R}^d} p_{r-s}(x,y)u^2(r,y)\varrho(drdy), \quad s \in [0,t], x \in \mathbb{R}^d. \tag{9.21}$$

If $J_t(s,x) := P_{t-s}\psi(x)$ for any $\psi \in C_b^+(\mathbb{R}^d)$, then we obtain in particular $u(s,x|t,J_t,\theta) = U_{s,t}(\theta\psi)(x)$ for all $s \in [0,t]$, $x \in \mathbb{R}^d$ and $\theta \in [0,\theta_t)$; recall that $(U_{s,t})$ denotes the log-Laplace semigroup associated with $\bar{X}$.

Proof (of Lemma 9.4) We first consider the sequence (a_n) of positive real numbers that is recursively defined as follows: $a_1 := c > 0$, $a_n := c\sum_{j=1}^{n-1} a_j a_{n-j}$ $(n \geq 2)$. By induction on n the terms of the sequence can be shown to be bounded as $a_n \leq c^{2n-1}4^n$ (the key is a clever rearrangement of the summands[34]). Now, define

$$C_{t,J_t} := \Big(\sup_{s\in[0,t]}\sup_{x\in\mathbb{R}^d} J_t(s,x)\Big) \vee \Big(\sup_{s\in[0,t]}\sup_{x\in\mathbb{R}^d} \frac{1}{2}\int_s^t \int_{\mathbb{R}^d} p_{r-s}(x,y)\varrho(drdy)\Big)$$

which is finite since $\varrho(dtdx)$ satisfies condition (B) (Lemmas 4.2$(i) \Rightarrow (ii)$ and 4.4(i) do trick here). By the definition of the $a_n(.,.|t,J_t)$ we easily obtain that $|a_n(s,x|t,J_t)| \leq a_n$ holds for all $n \geq 1$ where the a_n are defined as above with $c = C_{t,J_t}$. Therefore,

$$\sum_{n=1}^\infty |a_n(s,x|t,J_t)|\theta^n \leq \sum_{n=1}^\infty C_{t,J_t}^{2n-1}4^n\theta^n \qquad \forall s \in [0,t] \text{ and } x \in \mathbb{R}^d.$$

Hence, if we set $\theta_t = \theta_{t,J_t} := (4C_{t,J_t}^2)^{-1}$, the power series converges absolutely for all $\theta \in [0,\theta_t)$ uniformly in $s \in [0,t]$ and $x \in \mathbb{R}^d$. In particular, the map $(s,x) \mapsto u(s,x|t,J_t,\theta)$ provides a bounded measurable function on $[0,t]\times\mathbb{R}^d$ for every $\theta \in [0,\theta_t)$.

For brevity we now write $a_n(s,x)$ instead of $a_n(s,x|t,J_t)$. Then, for every $\theta \in [0,\theta_t)$, the function $(u(s,x|t,J_t,\theta) : s \in [0,t], x \in \mathbb{R}^d)$ solves the integral equation (9.21) since

$$\begin{aligned}
&\theta J_t(s,x) - \frac{1}{2}\int_s^t\int_{\mathbb{R}^d} p_{r-s}(x,y)u^2(r,y|t,J_t,\theta)\varrho(drdy)\\
&= a_1(s,x)\theta - \frac{1}{2}\int_s^t\int_{\mathbb{R}^d} p_{r-s}(x,y)\Big(\sum_{n=1}^\infty a_n(r,y)\theta^n\Big)^2\varrho(drdy)\\
&= a_1(s,x)\theta - \frac{1}{2}\int_s^t\int_{\mathbb{R}^d} p_{r-s}(x,y)\sum_{n=1}^\infty\Big[\sum_{j=1}^n a_j(r,y)a_{n+1-j}(r,y)\Big]\theta^{n+1}\varrho(drdy)\\
&= a_1(s,x)\theta + \sum_{n=1}^\infty\Big[-\frac{1}{2}\int_s^t\int_{\mathbb{R}^d} p_{r-s}(x,y)\Big(\sum_{j=1}^n a_j(r,y)a_{n+1-j}(r,y)\Big)\varrho(drdy)\Big]\ \theta^{n+1}\\
&= a_1(s,x)\theta + \sum_{n=2}^\infty\Big[-\frac{1}{2}\int_s^t\int_{\mathbb{R}^d} p_{r-s}(x,y)\Big(\sum_{j=1}^{n-1} a_j(r,y)a_{n-j}(r,y)\Big)\varrho(drdy)\Big]\ \theta^{n}\\
&= a_1(s,x)\theta + \sum_{n=2}^\infty a_n(s,x)\theta^n \ = \ \sum_{n=1}^\infty a_n(s,x)\theta^n \ = \ u(s,x|t,J_t,\theta).
\end{aligned}$$

The uniqueness of solutions can be shown by means of a Gronwall argument (Lemma 4.12) as in Step 5 of the proof of Theorem 7.3 (resp. 6.8). This completes the proof. □

Theorem 9.5 [MOMENTS] *For every $m \geq 1$, $\eta \in \mathcal{M}_f(\mathbb{R}^d)$, $0 \leq s \leq t$ and $\psi \in B_b(\mathbb{R}^d)$:*

$$\mathbb{E}_{s,\eta}\left[\langle \bar{X}_t,\psi\rangle^m\right] \ = \ m!\sum_{k=1}^m \frac{(-1)^{m+k}}{k!} \sum_{\substack{n_1,\ldots,n_k\geq 1\\ n_1+\cdots+n_k=m}} \prod_{i=1}^k \langle \eta, a_{n_i}(s,.|t,J_t)\rangle \tag{9.22}$$

[34] Axel Simroth showed me the detailed proof. For brevity we omit it here.

where the $a_n(.,.|t, J_t)$ are defined as in Lemma 9.4 with $J_t(s,x) := P_{t-s}\psi(x)$.

In the proof of Lemma 9.4 we have seen that $|a_n(s,x|t,J_t)| \le C_{t,J_t}^{2n-1} 4^n$ uniformly in $s \in [0,t]$ and $x \in \mathbb{R}^d$. So it is easy to see that all moments ($m \ge 1$) of the catalytic SBM are finite. In particular, we obtain for the first and the second moments:

$$\begin{aligned}
\mathbb{E}_{s,\eta}\left[\langle \bar{X}_t, \psi\rangle\right] &= \langle \eta, P_{t-s}\psi\rangle \qquad (9.23)\\
\mathbb{E}_{s,\eta}\left[\langle \bar{X}_t, \psi\rangle^2\right] &= \langle \eta, P_{t-s}\psi\rangle^2 + \int_{\mathbb{R}^d}\int_s^t\int_{\mathbb{R}^d} p_{r-s}(x,y)(P_{t-r}\psi)^2(y)\varrho(drdy)\,\eta(dx).
\end{aligned}$$

Proof (of Theorem 9.5) We first consider the case where the test function ψ is non-negative and continuous. Fix $t \ge 0$ and $\psi \in C_b^+(\mathbb{R}^d)$. For $s \in [0,t]$ and $m \ge 1$ we clearly have

$$\frac{\partial^m}{\partial\theta^m}\mathbb{E}_{s,\eta}\left[e^{-\langle \bar{X}_t, \theta\psi\rangle}\right]\Big|_{\theta=0+} = (-1)^m\,\mathbb{E}_{s,\eta}[\langle \bar{X}_t,\psi\rangle^m]. \qquad (9.24)$$

By Lemma 9.4 there is some $\theta_{t,\psi} > 0$ such that $u(s,x|t,\psi,\theta) := \sum_{n=1}^\infty a_n(s,x|t,J_t)\theta^n$ converges absolutely, uniformly in $s \in [0,t]$ and $x \in \mathbb{R}^d$, for all $\theta \in [0,\theta_t)$. Hence,

$$e^{-\langle \eta, u(s,.|t,J_t,\theta)\rangle} = e^{-\langle \eta, \sum_{n=1}^\infty a_n(s,.|t,J_t)\theta^n\rangle} = e^{-\sum_{n=1}^\infty \langle \eta, a_n(s,.|t,J_t)\rangle\theta^n} \qquad \forall \theta \in [0,\theta_{t,\psi}).$$

Then, using an induction on m and setting $\theta := 0$, we obtain for all $s \in [0,t]$ and $m \ge 1$:

$$\frac{\partial^m}{\partial\theta^m} e^{-\langle \eta, u(s,x|t,\psi,\theta)\rangle}\Big|_{\theta=0+} = m!\sum_{k=1}^m \frac{(-1)^k}{k!}\sum_{\substack{n_1,\dots,n_k\ge 1\\ n_1+\dots+n_k=m}}\prod_{i=1}^k \langle \eta, a_{n_i}(s,.|t,J_t)\rangle. \qquad (9.25)$$

Also, by (9.9) and $U_{s,t}(\theta\psi)(x) = u(s,x|t,J_t,\theta)$ for $J_t(s,x) = P_{t-s}\psi(x)$,

$$\mathbb{E}_{s,\eta}\left[e^{-\langle \bar{X}_t,\theta\psi\rangle}\right] = e^{-\langle \eta, u(s,.|t,\psi,\theta)\rangle} \qquad \forall\theta \in [0,\theta_{t,\psi}). \qquad (9.26)$$

Then (9.22) follows from (9.24), (9.25) and (9.26).

Now we intend to show that (9.22) remains true for signed continuous test functions ψ, i.e. for arbitrary $\psi \in C_b(\mathbb{R}^d)$. Let $\psi_+, \psi_- \ge 0$ be the positive, respectively negative, part of ψ. For $\lambda_+, \lambda_- \in \mathbb{R}$ set $\psi_{\lambda_+,\lambda_-} := \lambda_+\psi_+ + \lambda_-\psi_-$. Then the l.h.s. of (9.22) with ψ replaced by $\psi_{\lambda_+,\lambda_-}$ can be written as

$$\sum_{k=0}^m \binom{m}{k}\lambda_+^k\lambda_-^{m-k}\,\mathbb{E}_{s,\eta}\left[\langle \bar{X}_t,\psi_+\rangle^k\langle \bar{X}_t,\psi_-\rangle^{m-k}\right]. \qquad (9.27)$$

From the first part of the proof we know that $\mathbb{E}_{s,\eta}[\langle \bar{X}_t,\psi_+\rangle^{2k}]$ and $\mathbb{E}_{s,\eta}[\langle \bar{X}_t,\psi_-\rangle^{2(m-k)}]$ are finite for all k. Hence Hölder's inequality implies that the expectations on the r.h.s. of (9.27) are finite, too. That means $\mathbb{E}_{s,\eta}[\langle \bar{X}_t,\psi_{\lambda_+,\lambda_-}\rangle^m]$ is a polynomial in (λ_+,λ_-). By the recursive definition of the $a_n(.,.|t,J_t)$, the r.h.s. of (9.22) with ψ replaced by $\psi_{\lambda_+,\lambda_-}$ is a polynomial in (λ_+,λ_-) as well. From the first part of the proof we know that the l.h.s and the r.h.s. of (9.22) with ψ replaced by $\psi_{\lambda_+,\lambda_-}$ coincide for $\lambda_+,\lambda_- \ge 0$. But then, since the expressions on both sides are polynomials, they have to coincide for all $\lambda_+, \lambda_- \in \mathbb{R}$. Setting $\lambda_+ := 1$ and $\lambda_- := -1$ completes the proof for $\psi \in C_b(\mathbb{R}^d)$.

It remains to show (9.22) for general $\psi \in B_b(\mathbb{R}^d)$. For it we recall that $C_b(\mathbb{R}^d)$ is *bp*-dense in $B_b(\mathbb{R}^d)$; cf. [EK86], p.111. Thus, (9.22) can be inferred by means of a proper pointwise approximation, the recursive definition of the $a_n(.,.|t,J_t)$ and the dominated convergence theorem. □

The next result generalizes moment formula (9.22) of Theorem 9.5.

Theorem 9.6 [MOMENTS OF SUMS] *For all $m \geq 1$, $\eta \in \mathcal{M}_f(\mathbb{R}^d)$ and all $l \geq 1$, $0 \leq s \leq t_1 \leq \ldots \leq t_l =: t$ and $\psi_1, \ldots, \psi_l \in B_b(\mathbb{R}^d)$:*

$$\mathbb{E}_{s,\eta}\Big[\Big(\sum_{i=1}^{l} \langle \bar{X}_{t_i}, \psi_i \rangle\Big)^m\Big] \;=\; m! \sum_{k=1}^{m} \frac{(-1)^{m+k}}{k!} \sum_{\substack{n_1,\ldots,n_k \geq 1 \\ n_1+\cdots+n_k=m}} \prod_{i=1}^{k} \langle \eta, a_{n_i}(s,.|t,J_t)\rangle \tag{9.28}$$

where the $a_n(.,.|t,J_t)$ are defined as in Lemma 9.4 with $J_t(\tilde{s},x) := \sum_{i:t_i \geq \tilde{s}} P_{t_i-\tilde{s}}\psi_i(x)$.

Proof One can mimic the proof of Theorem 9.5. However, at one point one has to work a bit harder. In the proof of Theorem 9.5 we used the relation (9.26) which holds by the definition of $\bar{X}$. This time we need the analogue

$$\mathbb{E}_{s,\eta}\Big[e^{-\sum_{i=1}^{l}\langle \bar{X}_{t_i}, \theta\psi_i\rangle}\Big] = e^{-\langle \eta, u(s,.|t,J_t,\theta)\rangle} \qquad \forall \theta \in [0, \theta_{t,J_t}) \tag{9.29}$$

where $J_t(\tilde{s},x) := \sum_{i:t_i \geq \tilde{s}} P_{t_i-\tilde{s}}\psi_i(x)$. The validity of (9.29) is not obvious. However, it can be shown by an induction on l starting from (9.26) where the Markov property of $\bar{X}$ plays the central role. The same argument has been used in the proof of Lemme 4.3 of [Del96] for the case of time-constant catalysts (see also [LG99], Proposition II.7) and so, for brevity, we omit the details. □

For the following two corollaries we require $\bar{X}$ to be weakly continuous. Note that this demand is actually no restriction since, as we will see in the next section, the catalytic SBM possesses a modification which is weakly continuous, i.e. continuous w.r.t. the weak topology. The first corollary will be used for the construction of the collision measure of the catalytic SBM $\bar{X}$ and its catalyst $\varrho(dtdx)$ (Section 9.6). The second corollary is important for the characterization of $\bar{X}$ as solution to a certain martingale problem (Section 9.7).

Corollary 9.7 *Suppose $\bar{X}$ is continuous w.r.t. the weak topology. Then we have for every $m \geq 1$, $\eta \in \mathcal{M}_f(\mathbb{R}^d)$, $0 \leq s \leq t$, $\epsilon > 0$ and $f \in B_b([0,\infty) \times \mathbb{R}^d)$:*

$$\begin{aligned}\mathbb{E}_{s,\eta}\Big[\Big(\int_s^t \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(r,y)p_\epsilon(x,y)\bar{X}_r(dx)\varrho(drdy)\Big)^m\Big] \qquad\qquad\qquad\qquad & \qquad (9.30)\\ = m! \sum_{k=1}^{m} \frac{(-1)^{m+k}}{k!} \sum_{\substack{n_1,\ldots,n_k \geq 1 \\ n_1+\cdots+n_k=m}} \prod_{i=1}^{k} \langle \eta, a_{n_i}(s,.|t,J_t)\rangle &\end{aligned}$$

where the functions $a_n(.,.|t,J_t)$ are recursively defined as in Lemma 9.4 with $J_t(s,x) := \int_s^t \int_{\mathbb{R}^d} p_{r-s+\epsilon}(x,y)f(r,y)\varrho(drdy)$.

Proof We restrict our attention to those f for which $r \mapsto f(r,y)$ is uniformly continuous on $[s,t]$ uniformly in $y \in \mathbb{R}^d$.[35] This assumption on f guarantees that the map $(r,x) \mapsto f_{\epsilon,\varrho_1}(r,x) := \int_{\mathbb{R}^d} f(r,y)p_\epsilon(x,y)\varrho_1(r,dy)$ is uniformly continuous in $(r,x) \in [s,t] \times \mathbb{R}^d$. Indeed, using Lemma 4.2$(i) \Rightarrow (ii)$ and Lemma 4.5(ii),

$$
\begin{aligned}
&|f_{\epsilon,\varrho_1}(r,x) - f_{\epsilon,\varrho_1}(r',x')| \;\le\; |f_{\epsilon,\varrho_1}(r,x) - f_{\epsilon,\varrho_1}(r',x)| + |f_{\epsilon,\varrho_1}(r',x) - f_{\epsilon,\varrho_1}(r',x')| \\
&\quad\le\; \int_{\mathbb{R}^d} \sup_{z\in\mathbb{R}^d} |f(r,z) - f(r',z)| p_\epsilon(x,y)\varrho_1(r,dy) + \int_{\mathbb{R}^d} f(r',y)|p_\epsilon(x,y) - p_\epsilon(x',y)|\varrho_1(r,dy) \\
&\quad\le\; \sup_{z\in\mathbb{R}^d} |f(r,z) - f(r',z)| \frac{1}{(2\pi\epsilon)^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{2\epsilon}} \varrho_1(r,dy) \;+\; \|f\|_\infty \, c_t \, \epsilon^{d/2-1/-\beta_1/2}|x-x'| \\
&\quad\le\; \sup_{z\in\mathbb{R}^d} |f(r,z) - f(r',z)| \frac{1}{(2\pi\epsilon)^{d/2}} c_t(2\epsilon)^{\beta_1} \;+\; \|f\|_\infty \, c_t \, \epsilon^{d/2-1/-\beta_1/2}|x-x'| \\
&\quad\le\; c_{\epsilon,t,f}\Big(\sup_{z\in\mathbb{R}^d} |f(r,z) - f(r',z)| + |x-x'| \Big).
\end{aligned}
$$

Now, let $\pi = \{s = t_0 < t_1 < \cdots < t_{l_\pi} = t\}$ denote a partition of the interval $[s,t]$ and set $|\pi| := \max_{i=1,\ldots,l_\pi} |t_i - t_{i-1}|$. Further define for every $m \ge 1$:

$$
\begin{aligned}
Y^m &:= \Big(\int_s^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(r,y)p_\epsilon(x,y)\bar{X}_r(dx)\varrho(drdy) \Big)^m \;=\; \Big(\int_s^t \langle \bar{X}_r, f_{\epsilon,\varrho_1}(r,.)\rangle \varrho_2(dr) \Big)^m, \\
Y_\pi^m &:= \Big(\sum_{t_i \in\, \pi\setminus\{s\}} \langle \bar{X}_{t_i}, f_{\epsilon,\varrho_1}(t_i,.)\rangle \; \varrho_2((t_{i-1},t_i]) \Big)^m.
\end{aligned}
$$

Since $\bar{X}$ was assumed to be weakly continuous and $(r,x) \mapsto f_{\epsilon,\varrho_1}(r,x)$ was seen to be uniformly continuous on $[s,t] \times \mathbb{R}^d$, the map $r \mapsto \langle \bar{X}_r, f_{\epsilon,\varrho_1}(r,.)\rangle$ is $\mathbb{P}_{s,\eta}$-almost surely continuous on $[s,t]$. Therefore Y_π^m converges $\mathbb{P}_{s,\eta}$-almost surely to Y^m as $|\pi| \downarrow 0$ for every $m \ge 1$ (note that the Lebesgue-Stieltjes integral and the Riemann-Stieltjes integral coincide for continuous integrands). Further, since $(r,x) \mapsto f_{\epsilon,\varrho_1}(r,x)$ is clearly bounded, we get by Jensen's inequality and Theorem 9.5:

$$
\begin{aligned}
\mathbb{E}_{s,\eta}\left[(Y^m)^2\right] &\le \mathbb{E}_{s,\eta}\Big[\Big(\int_s^t \langle \bar{X}_r, \|f_{\epsilon,\varrho_1}\|_\infty\rangle \varrho_2([s,t]) \frac{\varrho_2(dr)}{\varrho_2([s,t])} \Big)^{2m}\Big] \\
&\le \|f_{\epsilon,\varrho_1}\|_\infty^{2m} \varrho_2([s,t])^{2m} \, \mathbb{E}_{s,\eta}\Big[\int_s^t \langle \bar{X}_r, \mathbf{1}\rangle^{2m} \frac{\varrho_2(dr)}{\varrho_2([s,t])} \Big] \\
&= \|f_{\epsilon,\varrho_1}\|_\infty^{2m} \varrho_2([s,t])^{2m-1} \int_s^t \mathbb{E}_{s,\eta}\left[\langle \bar{X}_r, \mathbf{1}\rangle^{2m}\right] \varrho_2(dr) \;<\; \infty,
\end{aligned}
$$

i.e. $Y^m \in L^2(\Omega, \mathbb{P}_{s,\eta})$ for every $m \ge 1$. Using Theorem 9.6 we can also check that the family $(Y_\pi^m : |\pi| \le 1)$ is $L^2(\Omega, \mathbb{P}_{s,\eta})$-bounded for every $m \ge 1$. Thus, $(Y_\pi^m : |\pi| \le 1)$ is uniformly $\mathbb{P}_{s,\eta}$-integrable for every $m \ge 1$ by Lemma 3.5. The $\mathbb{P}_{s,\eta}$-almost sure convergence of Y_π^m to $Y^m \in L^2(\Omega, \mathbb{P}_{s,\eta})$ together with the uniform $\mathbb{P}_{s,\eta}$-integrability of $(Y_\pi^m : |\pi| \le 1)$

[35] Once having proved (9.30) for such f's, the general case can be inferred by means of a proper pointwise approximation, the recursive definition of the $a_n(.,.|t,J_t)$ and the dominated convergence theorem.

imply $L^1(\Omega, \mathbb{P}_{s,\eta})$-convergence of Y^m_π to Y^m (cf. [Kal97], Proposition 3.12). In particular,

$$\begin{aligned}
&\mathbb{E}_{s,\eta}\Big[\Big(\int_s^t \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(r,y)p_\epsilon(x,y)\bar{X}_r(dx)\varrho(drdy)\Big)^m\Big] \qquad (9.31)\\
&= \lim_{|\pi|\downarrow 0} \mathbb{E}_{s,\eta}\Big[\Big(\sum_{t_i\in\ \pi\setminus\{s\}} \Big\langle \bar{X}_{t_i},\ \varrho_2((t_{i-1},t_i])\ f_{\epsilon,\varrho_1}(t_i,.)\Big\rangle\Big)^m\Big]\\
&= \lim_{|\pi|\downarrow 0} m! \sum_{k=1}^m \frac{(-1)^{m+k}}{k!} \sum_{n_1+\cdots+n_k=m} \prod_{i=1}^k \langle \eta, a_{n_i}(s,.|t,J_t^\pi)\rangle
\end{aligned}$$

where the $a_n(.,.|t,J_t^\pi)$ are defined as in Lemma 9.4 with

$$J_t^\pi(\tilde{s},x) := \sum_{t_i\in\ \pi\setminus\{s\},\ t_i\geq\tilde{s}} P_{t_i-\tilde{s}}(\varrho_2((t_{i-1},t_i])\ f_{\epsilon,\varrho_1}(t_i,.))(x)$$

and the second "=" is a consequence of (9.28). In order to complete the proof it suffices to show that the very r.h.s. of (9.31) coincides with the r.h.s. of (9.30). Since

$$\begin{aligned}
&J_t(\tilde{s},x) - J_t^\pi(\tilde{s},x)\\
&= \int_{\tilde{s}}^t \int_{\mathbb{R}^d} p_{r-\tilde{s}+\epsilon}(x,y)f(r,y)\varrho(drdy) - \sum_{t_i\in\ \pi\setminus\{s\},\ t_i\geq\tilde{s}} P_{t_i-\tilde{s}}(\varrho_2((t_{i-1},t_i])\ f_{\epsilon,\varrho_1}(t_i,.))(x)\\
&= \int_{\tilde{s}}^t (P_{r-\tilde{s}}f_{\epsilon,\varrho_1}(r,.))(x)\varrho_2(dr) - \sum_{t_i\in\ \pi\setminus\{s\},\ t_i\geq\tilde{s}} (P_{t_i-\tilde{s}}f_{\epsilon,\varrho_1}(t_i,.))(x)\ \varrho_2((t_{i-1},t_i])
\end{aligned}$$

bp-converges to 0 as $|\pi|\downarrow 0$, this can easily be concluded from the recursive definition of the $a_n(.,.|t,J_t)$ and $a_n(.,.|t,J_t^\pi)$. □

Corollary 9.8 [MOMENTS OF SUM-INTEGRAL MIXTURES] *Suppose $\bar{X}$ is continuous w.r.t. the weak topology. Then we have for every $m\geq 1$, $\eta\in\mathcal{M}_f(\mathbb{R}^d)$, $f\in B_b([0,\infty)\times\mathbb{R}^d)$, $0\leq s\leq v$ and all $l\geq 1$, $s\leq t_1\leq\ldots\leq t_l =: t$ and $\psi_1,\ldots,\psi_l\in B_b(\mathbb{R}^d)$:*

$$\begin{aligned}
&\mathbb{E}_{s,\eta}\Big[\Big(\sum_{i=1}^l \langle \bar{X}_{t_i},\psi_i\rangle + \int_s^v \langle \bar{X}_r, f(r,.)\rangle dr\Big)^m\Big] \qquad (9.32)\\
&\qquad = m! \sum_{k=1}^m \frac{(-1)^{m+k}}{k!} \sum_{\substack{n_1,\ldots,n_k\geq 1\\ n_1+\cdots+n_k=m}} \prod_{i=1}^k \langle \eta, a_{n_i}(s,.|t\vee v, J_{t\vee v})\rangle
\end{aligned}$$

where the functions $a_n(.,.|t\vee v, J_{t\vee v})$ are recursively defined as in Lemma 9.4 with $J_{t\vee v}(\tilde{s},x) := \sum_{i:t_i\geq\tilde{s}} P_{t_i-\tilde{s}}\psi_i(x) + \mathbf{1}_{v\geq\tilde{s}}\int_{\tilde{s}}^v (P_{r-\tilde{s}}f(r,.))(x)dr$.

Proof The proof is very similar to the proof of Corollary 9.7. So we omit it. □

9.5 Sample continuity and strong Markov property

In this section we show that the catalytic SBM $\bar{X}$ may be assumed to be continuous w.r.t. the weak topology (Theorem 9.11). Note that continuity of $t \mapsto \bar{X}_t(dx)$ w.r.t. the weak topology is equivalent to sequential continuity of $t \mapsto \bar{X}_t(dx)$ w.r.t. weak convergence. A main tool for the proof of Theorem 9.11 will be Theorem 9.6. As a consequence of the continuity we also obtain that $\bar{X}$ can be assumed to be a strong Markov process (Corollary 9.12). Note that sample continuity of the catalytic SBM with more special catalysts has been established earlier (cf. [Del96], [DF97]). We start with a crucial lemma.

Lemma 9.9 *Let $\varrho(dtdx) \in \mathcal{M}([0,\infty) \times \mathbb{R}^d)$ satisfy condition* (B) *with β_1, β_2. Fix $T > 0$, $\eta \in \mathcal{M}_f(\mathbb{R}^d)$, $n \geq 2$ and $\gamma \in (0, \beta)$ where $\beta := \beta_1/2 + \beta_2 - d/2$. Let $0 \leq t \leq t' \leq T$ and $a_n(.,.|T, J_T)$ be as in Lemma 9.4 with $J_T(s,x) := \mathbf{1}_{[0,t]}(s) P_{t-s}\psi(x) - \mathbf{1}_{[0,t']}(s) P_{t'-s}\psi(x)$ for some $\psi \in B_b(\mathbb{R}^d)$. Then there exists a constant $c_n = c_{T,\eta,n,\gamma} > 0$ (which is independent of t, t' and ψ) such that for all $s \in [0,T]$ and $x \in \mathbb{R}^d$:*

$$|a_n(s,x|T,J_T)| \leq c_n \|\psi\|_\infty^n |t - t'|^{n\gamma/2} \mathbf{1}_{[0,t']}(s). \tag{9.33}$$

If ψ is Lipschitz continuous, then (9.33) also holds for $n = 1$.

Proof We proceed by induction on n. For $n = 2$ we obtain for all $s \in [0,T]$ and $x \in \mathbb{R}^d$:

$$\begin{aligned}
&|a_2(s,x|T,J_T)| \\
&= \frac{1}{2} \int_s^T \int_{\mathbb{R}^d} p_{r-s}(x,y) \Big[\mathbf{1}_{[0,t]}(r) P_{t-r}\psi(y) - \mathbf{1}_{[0,t']}(r) P_{t'-r}\psi(y) \Big]^2 \varrho(drdy) \\
&\leq \int_s^T \int_{\mathbb{R}^d} p_{r-s}(x,y) \mathbf{1}_{[0,t]}(r) \Big[P_{t-r}\psi(y) - P_{t'-r}\psi(y) \Big]^2 \varrho(drdy) \\
&\qquad\qquad + \int_s^T \int_{\mathbb{R}^d} p_{r-s}(x,y) \mathbf{1}_{(t,t']}(r) (P_{t'-r}\psi)^2(y) \Big] \varrho(drdy) \\
&=: I_1(s,x) + I_2(s,x) \;=\; [I_1(s,x) + I_2(s,x)] \mathbf{1}_{[0,t']}(s).
\end{aligned}$$

Since $\varrho(dtdx)$ satisfy (B) we obtain by Lemma 4.2(i) $\Rightarrow$ (ii) and Lemma 4.4(i):

$$\begin{aligned}
I_2(s,x) &\leq \int_t^{t'} \int_{\mathbb{R}^d} p_{r-s}(x,y) \|\psi\|_\infty^2 \varrho(drdy) \\
&= \|\psi\|_\infty^2 \int_s^T \frac{1}{(2\pi(r-s))^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{2(r-s)}} \varrho_1(r,dy) \varrho_2(dr) \\
&\leq c_T \|\psi\|_\infty^2 \int_s^T \frac{1}{(r-s)^{d/2-\beta_1/2}} \varrho_2(dr) \\
&\leq c_T \|\psi\|_\infty^2 |t-t'|^{\beta_1/2+\beta_2-d/2} \;\leq\; c_T \|\psi\|_\infty^2 |t-t'|^\gamma \;=\; c_T \|\psi\|_\infty^2 |t-t'|^{2\gamma/2}.
\end{aligned}$$

By the same lemmas and Lemma 4.5(i) and $\gamma < \beta$ we also get

$$I_1(s,x) \;\leq\; \int_s^t \int_{\mathbb{R}^d} p_{r-s}(x,y) \Big[P_{t-r}\psi(y) - P_{t'-r}\psi(y) \Big]^2 \varrho(drdy)$$

$$
\begin{aligned}
&\leq \int_s^t \int_{\mathbb{R}^d} p_{r-s}(x,y)\Big[\int_{\mathbb{R}^d}\int_{t-r}^{t'-r} \frac{1}{u} p_{2u}(y,z)du\|\psi\|_\infty dz\Big]^2 \varrho(drdy)\\
&\leq c_T\|\psi\|_\infty^2 \int_s^t \frac{1}{(r-s)^{d/2}}\frac{1}{(t-r)^\gamma}\Big[\int_{t-r}^{t'-r}\frac{1}{u^{1-\gamma/2}}du\Big]^2 \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{2(r-s)}}\varrho_1(r,dy)\varrho_2(dr)\\
&\leq c_T'\|\psi\|_\infty^2 \int_s^t \frac{1}{(r-s)^{d/2}}\frac{1}{(t-r)^\gamma}\Big[|t-t'|^{\gamma/2}\Big]^2 (r-s)^{\beta_1/2}\varrho_2(dr)\\
&= c_T'\|\psi\|_\infty^2 |t-t'|^\gamma \int_s^t \frac{1}{(r-s)^{d/2-\beta_2}}\frac{1}{(t-r)^\gamma}\varrho_2(dr)\\
&\leq c_T'\|\psi\|_\infty^2|t-t'|^\gamma\Big[\frac{2^\gamma}{(t-s)^\gamma}\int_s^{s+\frac{t-2}{2}}\frac{1}{(r-s)^{d/2-\beta_1/2}}\varrho_2(dr)\\
&\qquad\qquad + \frac{2^{d/2-\beta_1/2}}{(t-s)^{d/2-\beta_1/2}}\int_{s+\frac{t-s}{2}}^t \frac{1}{(t-r)^\gamma}\varrho_2(dr)\Big]\\
&\leq c_T''\|\psi\|_\infty^2|t-t'|^\gamma\Big[\frac{2^\gamma}{(t-s)^\gamma}(t-s)^\beta + \frac{2^{d/2-\beta_1/2}}{(t-s)^{d/2-\beta_1/2}}(t-s)^{\beta_2-\gamma}\Big]\\
&\leq c_T'''\|\psi\|_\infty^2|t-t'|^\gamma \;=\; c_T'''\|\psi\|_\infty^2|t-t'|^{2\gamma/2}.
\end{aligned}
$$

This proves (9.33) for $n=2$. Now suppose (9.33) holds up to some $n\geq 2$. We show the step from n to $n+1$. By the definition of $a_{n+1}(.,.|T,J_T)$,

$$
\begin{aligned}
&|a_{n+1}(s,x|T,J_T)|\\
&\quad= \frac{1}{2}\int_s^T\int_{\mathbb{R}^d} p_{r-s}(x,y)\Big(\sum_{j=1}^n a_j(r,y|T,J_T)a_{n+1-j}(r,y|T,J_T)\Big)\varrho(drdy)\\
&\quad= \frac{1}{2}\sum_{j=1}^n\int_s^T\int_{\mathbb{R}^d} p_{r-s}(x,y)a_j(r,y|T,J_T)a_{n+1-j}(r,y|T,J_T)\varrho(drdy).
\end{aligned}
$$

For $j=1$ the summand can be bounded by

$$
\begin{aligned}
&\int_s^T\int_{\mathbb{R}^d} p_{r-s}(x,y)\Big[\mathbf{1}_{[0,t]}(r)P_{t-r}\psi(y)-\mathbf{1}_{[0,t']}(r)P_{t'-r}\psi(y)\Big]\Big(c_n\|\psi\|_\infty^n|t-t'|^{n\gamma/2}\Big)\varrho(drdy)\\
&\quad= c_n\|\psi\|_\infty^n|t-t'|^{n\gamma/2}\int_s^T\int_{\mathbb{R}^d} p_{r-s}(x,y)\Big[\mathbf{1}_{[0,t]}(r)P_{t-r}\psi(y)-\mathbf{1}_{[0,t']}(r)P_{t'-r}\psi(y)\Big]\varrho(drdy)\\
&\quad\leq c_n\|\psi\|_\infty^n c_T\|\psi\|_\infty|t-t'|^{n\gamma/2}|t-t'|^{2\gamma/2} \;\leq\; c_{n+1}'\|\psi\|_\infty^{n+1}|t-t'|^{(n+1)\gamma/2}
\end{aligned}
$$

where the first "$\leq$" can be obtained by proceeding as for the estimate of $a_2(.,.|T,J_T)$ (it does not matter whether considering $[...]^2$ or $[...]$). For $j\in\{2,\ldots,n-1\}$ the summand can be bounded by

$$
\begin{aligned}
&\int_s^T\int_{\mathbb{R}^d} p_{r-s}(x,y)a_j(r,y|T,J_T)a_{n+1-j}(r,y|T,J_T)\varrho(drdy)\\
&\quad\leq \int_s^T\int_{\mathbb{R}^d} p_{r-s}(x,y)\Big(c_j\|\psi\|_\infty^j|t-t'|^{j\gamma/2}\,c_{n+1-j}\|\psi\|_\infty^{n+1-j}|t-t'|^{(n+1-j)\gamma/2}\varrho(drdy)
\end{aligned}
$$

$$= \; c'_{n+1}\|\psi\|_\infty^{n+1}|t-t'|^{(n+1)\gamma/2}\int_s^T\int_{\mathbb{R}^d} p_{r-s}(x,y)\varrho(drdy) \;\le\; c''_{n+1}\|\psi\|_\infty^{n+1}|t-t'|^{(n+1)\gamma/2}$$

where the last "$\le$" is justified by Lemma 4.2$(i) \Rightarrow (ii)$ and Lemma 4.4(i). On the whole, $|a_{n+1}(s,x|T,J_T)| \le c_{n+1}\|\psi\|_\infty^{n+1}|t-t'|^{(n+1)\gamma/2}$ which completes the proof for $n \ge 2$.

The validity of (9.33) for $n = 1$ and Lipschitz continuous ψ is a consequence of Lemma 4.9. □

Lemma 9.10 *Let $\varrho(dtdx) \in \mathcal{M}([0,\infty)\times\mathbb{R}^d)$ satisfy condition* (B) *with β_1, β_2. Pick $s \ge 0$, $\eta \in \mathcal{M}_f(\mathbb{R}^d)$ and $\gamma \in (0,\beta)$, where $\beta := \beta_1/2 + \beta_2 - d/2$. Then for every $q \ge 1$, $T > s$, $\eta \in \mathcal{M}_f(\mathbb{R}^d)$ there exists a constant $c_q > 0$ (depending on q, γ, T and η) such that for all $t, t' \in [s,T]$ and Lipschitz continuous $\psi \in C_b(\mathbb{R}^d)$:*

$$\mathbb{E}_{s,\eta}\Big[\Big|\langle \bar{X}_t,\psi\rangle - \langle \bar{X}_{t'},\psi\rangle\Big|^{2q}\Big] \;\le\; c_q\|\psi\|_\infty^{2q}\,|t-t'|^{q\gamma}. \tag{9.34}$$

Proof Theorem 9.6 implies for $\psi \in C_b(\mathbb{R}^d)$, $0 \le t \le t' \le T$ and $m = 2q$ (with $q \ge 1$):

$$\mathbb{E}_{s,\eta}\Big[\Big|\langle \bar{X}_t,\psi\rangle - \langle \bar{X}_{t'},\psi\rangle\Big|^{m}\Big] = m!\sum_{k=1}^m \frac{(-1)^{m+k}}{k!}\sum_{n_1+\cdots+n_k=m}\prod_{i=1}^k\langle \eta, a_{n_i}(s,.|T,J_T)\rangle \tag{9.35}$$

where the $a_n(.,.|T,J_T)$ are recursively defined as in Lemma 9.4 with $J_T(\tilde{s},x) := \mathbf{1}_{[0,t]}(\tilde{s})P_{t-\tilde{s}}\psi(x) - \mathbf{1}_{[0,t']}(\tilde{s})P_{t'-\tilde{s}}\psi(x)$. If ψ is Lipschitz continuous, then we know from Lemma 9.9 that $|a_n(\tilde{s},x|T,J_T)| \le c_n\|\psi\|_\infty^n|t-t'|^{n\gamma/2}$ holds (uniformly in $\tilde{s} \in [0,T]$ and $x \in \mathbb{R}^d$) for all $n \ge 1$, where $c_n > 0$ depends on n, γ, T and is independent of t, t' and ψ. Hence, by (9.35) there exists for every $q \ge 1$ a constant $c_q > 0$ (depending on q, γ, T and η) such that (9.34) holds for all $t, t' \in [s,T]$ and all Lipschitz continuous $\psi \in C_b(\mathbb{R}^d)$. □

Now we turn to the sample continuity of the catalytic SBM.

Theorem 9.11 [SAMPLE CONTINUITY] *Let $\varrho(dtdx) \in \mathcal{M}([0,\infty)\times\mathbb{R}^d)$ satisfy condition* (B) *and pick $s \ge 0$ and $\eta \in \mathcal{M}_f(\mathbb{R}^d)$. Then the corresponding catalytic SBM $(\bar{X}_t : t \ge s)$ has a modification which is $\mathbb{P}_{s,\eta}$-almost surely continuous w.r.t. the weak topology.*

Proof Let $\hat{\mathbb{R}}^d = \mathbb{R}^d \cup \{\partial\}$ denote Alexandrov's one-point compactification of $\mathbb{R}^d$. The topology of $\hat{\mathbb{R}}^d$ is hence given by the topology of $\mathbb{R}^d$ united with the system $\{\hat{\mathbb{R}}^d \setminus K : K \subset \mathbb{R}^d \text{ compact}\}$. In particular, a sequence $(x_n) \subset \hat{\mathbb{R}}^d$ converges to ∂ if and only if for every compact $K \subset \mathbb{R}^d$ there exists some $n_K \ge 1$ such that $x_n \notin K$ for all $n \ge n_K$. We may and do impose a metric $\hat{d}$ on $\hat{\mathbb{R}}^d$ such that[36] $\hat{d}$ induces the same topology on $\hat{\mathbb{R}}^d$, $(\hat{\mathbb{R}}^d,\hat{d})$ is complete and separable, $\lim_{|x|\to\infty}\hat{d}(x,\partial) \to 0$ (where $x \in \mathbb{R}^d$) and there

[36]Such a metric always exists. To illustrate this for $d = 2$, let S_3 denote the 3-dimensional sphere (around the origin of $\mathbb{R}^2$). By means of the spherical projection $P : S_3 \to \mathbb{R}^2$ we obtain a one-to-one correspondence between $\mathbb{R}^2$ and $S_3 \setminus \mathsf{n}$ where n denotes the "north pole". Hence, when identifying n with ∂, $\hat{\mathbb{R}}^d$ and S_3 correspond one-to-one. Then $\hat{d}(x,y) := |P^{-1}(x) - P^{-1}(y)|$ $(x, y \in \hat{\mathbb{R}}^2)$ defines the wanted metric where $P^{-1}(\partial) := \mathsf{n}$. This construction also works for $\mathbb{R}^d$ (and the $(d+1)$-dimensional sphere) for any $d \ge 1$.

exists some constant $c > 0$ with $\hat{d}(x,y) \leq c|x-y|$ for all $x, y \in \mathbb{R}^d$. In particular, by Proposition 2.10 there exists a countable sequence $\{f_k\} \subset C_b(\hat{\mathbb{R}}^d) = C_c(\hat{\mathbb{R}}^d)$ such that the complete and separable metric $d_{\mathcal{M}(\hat{\mathbb{R}}^d)}$, defined in (2.4), generates the weak topology on $\mathcal{M}_f(\hat{\mathbb{R}}^d) = \mathcal{M}(\hat{\mathbb{R}}^d)$ (note that the vague and the weak topology on $\mathcal{M}_f(\hat{\mathbb{R}}^d) = \mathcal{M}(\hat{\mathbb{R}}^d)$ coincide since $\hat{\mathbb{R}}^d$ is compact). Note that $\psi \in C_b(\hat{\mathbb{R}}^d) = C(\hat{\mathbb{R}}^d)$ if and only if $\psi|_{\mathbb{R}^d} \in C_b(\mathbb{R}^d)$ and $\lim_{|x|\to\infty} \psi(x) = \psi(\partial)$. The space $C_{Lip}(\hat{\mathbb{R}}^d)$ of $(\hat{d}, |.|)$-Lipschitz continuous functions on $\hat{\mathbb{R}}^d$ is $\|.\|_\infty$-dense in $C_b(\hat{\mathbb{R}}^d) = C(\hat{\mathbb{R}}^d)$. Furthermore one can find a countable subset $\{\tilde{f}_k\} \subset C_{Lip}(\hat{\mathbb{R}}^d)$ which is $\|.\|_\infty$-dense in $C_{Lip}(\hat{\mathbb{R}}^d)$ and so in $C_b(\hat{\mathbb{R}}^d)$. That is, the set $\{f_k\}$ may be assumed to consist only of functions from $C_{Lip}(\hat{\mathbb{R}}^d)$ (cf. the discussion immediately after Proposition 2.10). In particular, $f_k|_{\mathbb{R}^d}$ can be assumed to be $(|.|, |.|)$-Lipschitz continuous[37] for every $k \geq 1$ since $\psi \in C_{Lip}(\hat{\mathbb{R}}^d)$ clearly implies Lipschitz continuity for $\psi|_{\mathbb{R}^d}$ (recall $\hat{d}(x,y) \leq c|x-y|$).

Step 1. We first show that $(\bar{X}_t : t \geq s)$, regarded as an $\mathcal{M}_f(\hat{\mathbb{R}}^d)$-valued process (i.e. $X_t(\{\partial\}) := 0$ for all $t \geq s$), possesses a weakly continuous modification. Choose $q \geq 1$ in such a manner that $q\gamma > 1$ for some $\gamma \in (0, \beta_1/2 + \beta_2 - d/2)$. Using techniques as in the proof of Proposition 3.8 and $\bar{X}_t(\{\partial\}) = 0$ $(\forall t \geq s)$, it can be deduced from Lemma 9.10 that the real-valued processes $t \mapsto \langle \bar{X}_t, f_k \rangle$ $(k \geq 1)$ are uniformly continuous on $D_{s,T} := \cup_{l \geq 1}((2^{-l}\mathbb{Z}) \cap [s,T])$, $\mathbb{P}_{s,\eta}$-almost surely. Consequently, $t \mapsto \bar{X}_t(dx)$ is $\mathbb{P}_{s,\eta}$-almost surely uniformly continuous on $D_{s,T}$ w.r.t. $d_{\mathcal{M}(\hat{\mathbb{R}}^d)}$. If Ω_0 denotes the exceptional null set, define $\bar{X}''(\omega) :\equiv 0$ for $\omega \in \Omega_0$, and

$$\bar{X}''_t(\omega, dx) := \begin{cases} \bar{X}_t(\omega, dx) & , \quad t \in D_{s,T} \\ \lim_{D_{s,T} \ni r \to t} \bar{X}_r(\omega, dx) & , \quad \text{otherwise} \end{cases}$$

for $\omega \notin \Omega_0$. Here the limit is taken w.r.t. $d_{\mathcal{M}(\hat{\mathbb{R}}^d)}$ and exists since $[\mathcal{M}_f(\hat{\mathbb{R}}^d), d_{\mathcal{M}(\hat{\mathbb{R}}^d)}]$ is complete. The process $(\bar{X}''_t : t \in [s,T])$ is clearly weakly continuous, and even uniformly continuous w.r.t. $d_{\mathcal{M}(\hat{\mathbb{R}}^d)}$. Also, with help of (9.34) one can show that

$$\mathbb{P}_{s,\eta}\Big[\langle \bar{X}''_t, f_k \rangle = \langle \bar{X}''_t, f_k \rangle \text{ for all } k \geq 1\Big] = 1 \qquad \forall t \in [s,T]$$

holds. So $\mathbb{P}_{s,\eta}[\bar{X}_t(dx) = \bar{X}''_t(dx)] = 1$ for all $t \in [s,T]$ since $\{f_k\}$ is separating in $\mathcal{M}_f(\hat{\mathbb{R}}^d)$. Hence $(\bar{X}''_t : t \in [s,T])$ is also a modification of $(\bar{X}_t : t \in [s,T])$. Then it is easy to construct a modification $(\bar{X}'_t : t \geq s)$ of $(\bar{X}_t : t \geq s)$ which is weakly continuous on $[s, \infty)$. Note that $(\bar{X}'_t : t \geq s)$ is $\mathcal{M}_f(\hat{\mathbb{R}}^d)$-valued and not yet known to take values in $\mathcal{M}_f(\mathbb{R}^d)$ for all times $t \geq s$, $\mathbb{P}_{s,\eta}$-almost surely (although we already know that it takes values in $\mathcal{M}_f(\mathbb{R}^d)$ on any fixed countable subset of $[s,\infty)$, $\mathbb{P}_{s,\eta}$-almost surely, since $\bar{X}'$ is a modification of $\bar{X}$).

Step 2. We next show that $(\bar{X}'_t : t \geq s)$ takes values in $\mathcal{M}_f(\mathbb{R}^d)$ for all times $t \geq s$, $\mathbb{P}_{s,\eta}$-almost surely. The key is the fact that $(\langle \bar{X}, P_{T-t}\psi \rangle : t \in [s,T])$ is a martingale for every $C_b^2(\mathbb{R}^d)$ and $T > s$ (cf. Remark 9.17 below). Pick a sequence $(\psi_n) \subset C_b^2(\mathbb{R}^d)$ such that $0 \leq \psi_n \leq 1$, $\psi_n = 1$ on $\mathbb{R}^d \setminus [-n,n]^d$ and $\psi_n = 0$ on $[-(n-1), n-1]^d$ for every $n \geq 1$. Note that $\lim_{|x|\to\infty} P_v\psi_n(x) = 1$ for all $n \geq 1$ and $v \geq 0$. In particular, if we extend $P_v\psi_n$

[37]This is the reason why we work with the compactification $\hat{\mathbb{R}}^d$ of $\mathbb{R}^d$. The Lipschitz continuity of the f_k is necessary for the application of Lemma 9.10.

from $\mathbb{R}^d$ to $\hat{\mathbb{R}}^d$ by setting $P_v\psi_n(\partial) := 1$, we have $P_v\psi_n \in C_b(\hat{\mathbb{R}}^d)$ and $\mathbf{1}_{\{\partial\}} \leq P_v\psi_n$ for all $n \geq 1$ and $v \geq 0$. By Doob's inequality (Proposition 3.20) and the fact that $\bar{X}'$ is a modification of $\bar{X}$, we obtain for every $T > s$, $\epsilon > 0$ and $n \geq 1$:

$$
\begin{aligned}
&\mathbb{P}_{s,\eta}\Big[\sup_{t\in[s,T]} \langle \bar{X}'_t, \mathbf{1}_{\{\partial\}}\rangle \geq \epsilon\Big] \\
&\quad\leq\ \mathbb{P}_{s,\eta}\Big[\sup_{t\in[s,T]} \langle \bar{X}'_t, P_{T-t}\psi_n\rangle \geq \epsilon\Big] \ =\ \mathbb{P}_{s,\eta}\Big[\sup_{t\in D_{s,T}} \langle \bar{X}'_t, P_{T-t}\psi_n\rangle \geq \epsilon\Big] \\
&\quad=\ \mathbb{P}_{s,\eta}\Big[\sup_{t\in D_{s,T}} \langle \bar{X}_t, P_{T-t}\psi_n\rangle \geq \epsilon\Big] \ \leq\ \mathbb{P}_{s,\eta}\Big[\sup_{t\in[s,T]} \langle \bar{X}_t, P_{T-t}\psi_n\rangle \geq \epsilon\Big] \\
&\quad\leq\ \epsilon^{-1}\,\mathbb{E}_{s,\eta}[\langle \bar{X}_T, \psi_n\rangle] \ =\ \epsilon^{-1}\,\langle \eta, P_{T-s}\psi_n\rangle.
\end{aligned}
$$

The latter bound vanishes as $n \to \infty$ since $\eta(dx)$ is finite. Thus the l.h.s. vanishes for every $\epsilon > 0$ (and $T > s$) and so $\mathbb{P}_{s,\eta}[\sup_{t\in[s,\infty)} \langle \bar{X}'_t, \mathbf{1}_{\{\partial\}}\rangle = 0] = 1$. That means $(\bar{X}'_t : t \geq s)$ takes indeed values in $\mathcal{M}_f(\mathbb{R}^d)$ for all times $t \geq s$, $\mathbb{P}_{s,\eta}$-almost surely.

Step 3. In order to complete the proof we have to show yet that $\bar{X}'$ is also continuous w.r.t. the weak topology on $\mathcal{M}_f(\mathbb{R}^d)$; so far we only know that it is continuous w.r.t. the weak topology on $\mathcal{M}_f(\hat{\mathbb{R}}^d)$. From Steps 1 and 2 we deduce that $\lim_{|t-t'|\downarrow 0} |\langle \bar{X}'_t, \psi\rangle - \langle \bar{X}'_{t'}, \psi\rangle| = 0$ holds for all $\psi \in \hat{C}_b(\mathbb{R}^d) := \{\phi|_{\mathbb{R}^d} : \phi \in C_b(\hat{\mathbb{R}}^d)\}$. Since $C_c(\mathbb{R}^d) \cup \{\mathbf{1}\}$ is contained in $\hat{C}_b(\mathbb{R}^d)$ and is also weak convergence determining in $\mathcal{M}_f(\mathbb{R}^d)$, we obtain the desired continuity w.r.t. the weak topology on $\mathcal{M}_f(\mathbb{R}^d)$. $\square$

Corollary 9.12 [STRONG MARKOV PROPERTY] *The catalytic SBM $\bar{X}$ with catalyst $\varrho(dtdx) \in \mathcal{M}([0,\infty)\times\mathbb{R}^d)$ satisfying condition* (B) *can be defined as a (canonical) continuous strong Markov process.*

Proof In view of Theorem 9.11, we may define the catalytic SBM $\bar{X}$ as a canonical *continuous* Markov process (recall that $\mathbb{P}_{s,\Upsilon}[.] = \int \mathbb{P}_{s,\eta}[.]\Upsilon(d\eta)$ holds for every $\Upsilon \in \mathcal{M}_1(\mathcal{M}_f(\mathbb{R}^d))$). Hence, by Proposition 3.51, it suffices to show that $[0,t) \ni s \mapsto U_{s,t}\psi$ is right-continuous w.r.t. $\|.\|_\infty$ for every $t > 0$ and $\psi \in C_b^+(\mathbb{R}^d)$. From the definition of $U_{s,t}\psi(x)$ we know that $|U_{s,t}\psi(x) - U_{s',t}\psi(x)|$ is bounded by

$$|P_{t-s}\psi(x) - P_{t-s'}\psi(x)| + \frac{1}{2}\int_{s\wedge s'}^t \int_{\mathbb{R}^d} |p_{r-s}(x,y) - p_{r-s'}(x,y)|\, \|(U_{r,t}\psi)^2(.)\|\, \varrho(drdy)$$

(where $p_r :\equiv 0$ for $r < 0$). With help of Lemma 4.9 as well as Lemma 4.7 and the finiteness of $\sup_{r\in[0,t]} \|(U_{r,t}\psi)^2(.)\|_\infty$ we conclude that for every $t_0 \in (0,t)$:

$$|U_{s,t}\psi(x) - U_{s',t}\psi(x)| \leq c_{t_0,\psi}|s-s'|^{\beta_2} \qquad \forall s,s' \in [0,t_0] \quad (\text{uniformly in } x \in \mathbb{R}^d).$$

That is, $[0,t) \ni s \mapsto U_{s,t}\psi$ is even continuous w.r.t. $\|.\|_\infty$. $\square$

9.6 Collision measure of catalyst and reactant

This section is devoted to the collision measure of the catalytic SBM $\bar{X}$ and its catalyst $\varrho(dtdx)$. As before, we assume the catalyst to satisfy condition (B) and require $\bar{X}$ to be weakly continuous (which is no significant restriction, cf. Theorem 9.11). Here the collision measure – denoted by $C_{[\bar{X},\varrho]}(dtdx)$ – is regarded as the collision measure of $\mu(dtdx) := \varrho(dtdx)$ and $\mu'(dtdx) := \bar{X}_t(dx)dt$ in the sense of (8.7). It will play an important role in the next section where we characterize the catalytic SBM as the unique solution to a certain martingale problem. Set $H_b([s,\infty)\times\mathbb{R}^d) := \cup_{T>s} B_b^T([s,\infty)\times\mathbb{R}^d)$ where $B_b^T([s,\infty)\times\mathbb{R}^d)$ denote the space of bounded measurable functions f on $[s,\infty)\times\mathbb{R}^d$ with $f(t,.)\equiv 0$ for all $t>T$. The accurate definition of the collision measure is:

Definition 9.13 [COLLISION MEASURE] *Let $(\bar{Y}_t(dx) : t \geq s)$ be an $\mathcal{M}_f(\mathbb{R}^d)$-valued process with initial state $\bar{Y}_s = \eta \in \mathcal{M}_f(\mathbb{R}^d)$ on some probability space $[\Omega,\mathcal{F},\mathbb{P}]$. A random measure $C_{[\bar{Y},\varrho]}(dtdx)$ on $[s,\infty)\times\mathbb{R}^d$ is called collision measure of $\bar{Y}$ and $\varrho(dtdx)$ if for every $f \in H_b([s,\infty)\times\mathbb{R}^d)$ the following two assertions hold:*

(i) $(\int_s^t\int_{\mathbb{R}^d} f(r,y)C_{[\bar{Y},\varrho]}(drdy) : t \geq s)$ *is* $(\mathcal{F}^{\bar{Y}}_{[s,t]})$*-adapted and* $\mathbb{P}$*-a.s. continuous,*

(ii) $\int_s^\infty\int_{\mathbb{R}^d} f(t,x)C_{[\bar{Y},\varrho]}(dtdx) = \lim_{\epsilon\downarrow 0}\int_s^\infty\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(t,x)p_\epsilon(x,x')\bar{Y}_t(dx')\varrho(dtdx)$ $\mathbb{P}$*-a.s.*

Note that the notion of collision measures (resp. collision local times) of two measure-valued processes already appeared in [BEP91]. While $\mathbb{P}$-almost sure uniqueness of the collision measure is ensured by part (ii) of the definition, existence might fail. For $\bar{X}$ and ϱ, however, $C_{[\bar{X},\varrho]}$ does exist:

Theorem 9.14 [COLLISION MEASURE OF CATALYST AND REACTANT] *Let $\varrho(dtdx)$ satisfy condition* (B)*. Then, for every $s \geq 0$ and $\eta \in \mathcal{M}_f(\mathbb{R})$, the collision measure $C_{[\bar{X},\varrho]}(dtdx)$ of the weakly continuous catalytic SBM $(\bar{X}_t : t \geq s)$ and its catalyst $\varrho(dtdx)$ exists under $\mathbb{P}_{s,\eta}$. Moreover, for every $m \geq 1$, $T > s$ and $f \in B_b^T([s,\infty)\times\mathbb{R}^d)$:*

$$\begin{aligned}&\mathbb{E}_{s,\eta}\Big[\Big(\int_s^\infty\int_{\mathbb{R}^d} f(t,x)C_{[\bar{X},\varrho]}(dtdx)\Big)^m\Big] \qquad (9.36)\\ &\qquad = m!\sum_{k=1}^m\frac{(-1)^{m+k}}{k!}\sum_{n_1+\cdots+n_k=m}\prod_{i=1}^k\langle\eta, a_{n_i}(s,.|T,J)\rangle\end{aligned}$$

where the functions $a_n(.,.|T,J_T)$ are recursively defined as in Lemma 9.4 with $J_T(s,x) := \int_s^T\int_{\mathbb{R}^d} p_{r-s}(x,y)f(r,y)\varrho(drdy)$.

In particular we obtain for the first moment:

$$\mathbb{E}_{s,\eta}\Big[\int_s^\infty\int_{\mathbb{R}^d} f(t,x)C_{[\bar{X},\varrho]}(dtdx)\Big] = \int_{\mathbb{R}^d}\int_s^\infty\int_{\mathbb{R}^d} p_{r-s}(x,y)f(t,y)\varrho(drdy)\,\eta(dx). \qquad (9.37)$$

For the proof of Theorem 9.14 we borrow ideas of Delmas ([Del96]) who studied the case of time-constant catalysts. Let us define the approximate collision measure $C^\epsilon_{[\bar{X},\varrho]}$ by

$$C^\epsilon_{[\bar{X},\varrho]}(dtdx) := \int_{\mathbb{R}^d} p_\epsilon(x,x')\bar{X}_t(dx')\varrho(dtdx). \qquad (9.38)$$

The crux of the proof of Theorem 9.14 is the following lemma.

Lemma 9.15 *Let $\varrho(dtdx)$ satisfy condition* (B) *with β_1, β_2 and set $\beta := \beta_1/2 + \beta_2 - d/2$. Pick $0 \le s < T$ and $\eta \in \mathcal{M}_f(\mathbb{R}^d)$. Then we obtain for every $f \in B_b^T([s,\infty) \times \mathbb{R}^d)$, $\epsilon, \epsilon' \in (0,1]$, $t, t' \in [s,T]$ and $q \ge 1$:*

$$\mathbb{E}_{s,\eta}\Big[\Big|\langle C^{\epsilon}_{[\bar{X},\varrho]}, f\rangle - \langle C^{\epsilon'}_{[\bar{X},\varrho]}, f\rangle\Big|^{2q}\Big] \;\le\; c_{q,T}\,\|f\|_\infty^{2q}\,|\epsilon - \epsilon'|^{2q\beta} \tag{9.39}$$

$$\mathbb{E}_{s,\eta}\Big[\Big|\langle C^{\epsilon}_{[\bar{X},\varrho]}, \mathbf{1}_{[s,t]}f\rangle - \langle C^{\epsilon}_{[\bar{X},\varrho]}, \mathbf{1}_{[s,t']}f\rangle\Big|^{2q}\Big] \;\le\; c_{q,T}\,\|f\|_\infty^{2q}\,|t - t'|^{2q\beta}. \tag{9.40}$$

Proof By Corollary 9.7 we have

$$\begin{aligned}
&\mathbb{E}_{s,\eta}\Big[\Big|\langle C^{\epsilon}_{[\bar{X},\varrho]}, f\rangle - \langle C^{\epsilon'}_{[\bar{X},\varrho]}, f\rangle\Big|^{2q}\Big] \qquad (9.41)\\
&= \mathbb{E}_{s,\eta}\Big[\Big(\int_s^\infty \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(r,x)\,\big(p_\epsilon(x,x') - p_{\epsilon'}(x,x')\big)\,\bar{X}_r(dx')\varrho(drdx)\Big)^{2q}\Big]\\
&= (2q)!\sum_{k=1}^{2q}\frac{(-1)^{2q+k}}{k!}\sum_{n_1+\cdots+n_k=2q}\prod_{i=1}^{k}\langle \eta, a_{n_i}(s,.|T,J_T)\rangle
\end{aligned}$$

where the functions $a_n(.,.|T,J_T)$ are recursively defined as in Lemma 9.4 with

$$J_T(s,x) := \int_s^T \int_{\mathbb{R}^d} (p_{r-s+\epsilon}(x,y) - p_{r-s+\epsilon'}(x,y)) f(r,y)\varrho(drdy).$$

By means of Lemma 4.8 we obtain $|J_T(s,x)| \le c_T\|f\|_\infty|\epsilon - \epsilon'|^\beta$ uniformly in $s \in [0,T]$ and $x \in \mathbb{R}^d$. Then (9.39) follows from (9.41) by the recursive definition of the $a_n(.,.|T,J_T)$. Inequality (9.40) can be shown similarly. We omit the details. □

Proof (of Theorem 9.14) For brevity we write C and C^ϵ instead of $C_{[\bar{X},\varrho]}$, respectively $C^\epsilon_{[\bar{X},\varrho]}$. ($C^\epsilon_{[\bar{X},\varrho]}$ was defined in (9.38)). We proceed in three steps:

Step 1. For any $f \in H_b([s,\infty) \times \mathbb{R}^d)$ set $\tilde{C}^\epsilon(f) := \langle C^\epsilon, f\rangle$, $\tilde{C}(f) := \liminf_{\epsilon\downarrow 0} \tilde{C}^\epsilon(f)$ and choose $T = T(f) > s$ in such a manner that $f \in B_b^T([s,\infty)\times\mathbb{R}^d)$ holds . The process $(\tilde{C}^\epsilon(f) : \epsilon \in (0,1])$ is easily seen to be continuous. By (9.39) of Lemma 9.15 (together with arguments as in the proof of Lemma 3.8) it is even Hölder continuous. In particular, $\tilde{C}(f) = \lim_{\epsilon\downarrow 0}\tilde{C}^\epsilon(f)$ $\mathbb{P}_{s,\eta}$-almost surely. By (9.39) of Lemma 9.15 we also know that $\tilde{C}(f) = \lim_{\epsilon\downarrow 0}\tilde{C}^\epsilon(f)$ holds in $L^{2q}(\mathbb{P}_{s,\eta})$ $\forall q \ge 1$ and so in $L^m(\mathbb{P}_{s,\eta})$ $\forall m \ge 1$. Further, for every $m \ge 1$ we obtain by Corollary 9.7 that the moment $\mathbb{E}_{s,\eta}[(\tilde{C}^\epsilon(f))^m]$ equals the r.h.s. of (9.36) with $a_n(.,.|T,J_T)$ replaced by $a_n(.,.|T,J_T^\epsilon)$ where

$$J_T^\epsilon(s,x) := \int_s^T \int_{\mathbb{R}^d} p_{r-s+\epsilon}(x,y) f(r,y)\varrho(drdy).$$

By Lemma 4.8 we have $|J_T(s,x) - J_T^\epsilon(s,x)| \le c_T\|f\|_\infty|\epsilon - \epsilon'|^\beta$ uniformly in $s \in [0,T]$ and $x \in \mathbb{R}^d$, where β is defined as in Lemma 9.15. Taking the recursive definition of the $a_n(.,.|T,J_T)$, respectively $a_n(.,.|T,J_T^\epsilon)$, into account it is not hard to deduce that r.h.s. of (9.36) with $a_n(.,.|T,J_T)$ replaced by $a_n(.,.|T,J_T^\epsilon)$ converges to the r.h.s. of (9.36) as $\epsilon \downarrow 0$.

Combining this with the $L^m(\mathbb{P}_{s,\eta})$-convergence of $\tilde{C}^\epsilon(f)$ to $\tilde{C}(f)$ yields that $\mathbb{E}_{s,\eta}[(\tilde{C}(f))^m]$ equals the r.h.s. of (9.36).

Step 2. Next we show that there exists a random measure $C(dtdx)$ on $[s,\infty)\times\mathbb{R}^d$ satisfying $\langle C,f\rangle = \tilde{C}(f)$ $\mathbb{P}_{s,\eta}$-almost surely, for every $f\in H_b([s,\infty)\times\mathbb{R}^d)$. We intend an application of Proposition 3.33. For the moment fix $N\geq 1$. Set $S_N := [s{+}N{-}1, s{+}N)\times\mathbb{R}^d$ as well as $\Xi_N^\epsilon(\psi) := \tilde{C}^\epsilon(f_{N,\psi})$ and $\Xi_N(\psi) := \tilde{C}(f_{N,\psi})$ for all $\psi\in B_b(S_N)$, where $f_{N,\psi} := \psi$ (resp. $f_{N,\psi} := 0$) on S_N (resp. on S_N^c). Ξ_N^ϵ and Ξ_N are mappings from $B_b(S_N)$ to $B_b(\Omega)$. The functionals Ξ_N^ϵ clearly satisfy (i) and (ii) of Proposition 3.33 and so does Ξ_N. Ξ_N also satisfies (iii) of Proposition 3.33. Indeed, pick $(\psi_k)\subset B_b^+(S_N)$ such that $\psi_k\uparrow\psi\in B_b^+(S_N)$. Then $(\Xi_N^\epsilon(\psi_k))_{k\geq 1}$ is $\mathbb{P}_{s,\eta}$-almost surely non-decreasing and dominated by $\Xi_N^\epsilon(\psi)$. It follows that $(\Xi_N(\psi_k))_{k\geq 1}$ is $\mathbb{P}_{s,\eta}$-almost surely non-decreasing and dominated by $\Xi_N(\psi)$. Moreover, from Step 1 we known that

$$\mathbb{E}_{s,\eta}[|\Xi_N(\psi_k)-\Xi_N(\psi)|^2] = \langle\eta, J_N(s,.)\rangle^2 + \int_{\mathbb{R}^d}\int_{S_N} p_{r-s}(x,y)(J_N(r,y))^2\varrho(drdy)\,\eta(dx)$$

holds for

$$J_N(s,x) := \int_{S_N} p_{r-s}(x,y)[\psi_k(r,y)-\psi(r,y)]\varrho(drdy).$$

By means of dominated convergence we conclude that $\mathbb{E}_{s,\eta}[|\Xi_N(\psi_k)-\Xi_N(\psi)|^2]$ converges to 0 as $\psi_k\uparrow\psi$. In particular, $\Xi_N(\psi_k)$ converges $\mathbb{P}_{s,\eta}$-almost surely to $\Xi_N(\psi)$ along a suitable subsequence of (ψ_k). However, since $(\Xi_N(\psi_k))_{k\geq 1}$ is non-decreasing and dominated by $\Xi_N(\psi)$, it even converges $\mathbb{P}_{s,\eta}$-almost surely to $\Xi_n(\psi)$ along (ψ_k), i.e. (iii) holds.

Since Ξ_N satisfies $(i)-(iii)$ of Proposition 3.33, there exists a finite random measure $C_N(dtdx)$ on S_N satisfying $\langle C_N,\psi\rangle = \Xi_N(\psi)$ $\mathbb{P}_{s,\eta}$-almost surely, for every $\psi\in B_b(S_N)$. Then $\langle C,f\rangle := \sum_{N=1}^\infty\langle C_N,\mathbf{1}_{S_N}f\rangle$, $f\in H_b([s,\infty)\times\mathbb{R}^d)$, defines a random measure $C(dtdx)$ on $[s,\infty)\times\mathbb{R}^d$ satisfying (ii) of Definition 9.13 as well as moment formula (9.36).

Step 3. It remains to show (i) of Definition 9.13. The $(\mathcal{F}_{[s,t]}^{\bar{X}})$-adaption of the process $(\int_s^t\int_{\mathbb{R}^d} f(r,y)C(drdy) : t\geq s)$ is an immediate consequence of the definition of $\tilde{C}$. By means of (9.40) and Fatou's lemma we further obtain for every $T>s$ and all $t,t'\in[s,T]$ and $q\geq 1$:

$$\mathbb{E}_{s,\eta}\Big[\Big|\langle C,\mathbf{1}_{[s,t]}f\rangle - \langle C,\mathbf{1}_{[s,t']}f\rangle\Big|^{2q}\Big] \leq c_{q,T}\,\|f\|_\infty^{2q}\,|t-t'|^{2q\beta}.$$

Using techniques as in the proof of Proposition 3.8 one can deduce for every $T>s$ that the process $(\int_s^t\int_{\mathbb{R}^d} f(r,y)C(drdy) : t\in[s,T])$ is $\mathbb{P}_{s,\eta}$-almost surely continuous on $D_{s,T} := \cup_{l\geq 1}((2^{-l}\mathbb{Z})\cap[s,T])$. However, the process $(\int_s^t\int_{\mathbb{R}^d} f(r,y)C(drdy) : t\geq s)$ is also non-decreasing, and so it is $\mathbb{P}_{s,\eta}$-almost surely continuous on $[s,\infty)$. □

9.7 Martingale problem

In this section we intend to characterize the catalytic SBM $\bar{X}$ as unique solution to a certain martingale problem (cf. Definition 9.20, Theorem 9.21). As an easy consequence of the martingale characterization we obtain a representation of $\bar{X}$ as solution to a certain stochastic integral equation (sometimes called *Green's function representation*, cf.

Corollary 9.22). We again assume that the catalyst $\varrho(dtdx)$ satisfies condition (B). Let $C_b^{1,2}([s,\infty)\times\mathbb{R}^d)$ denote the space of functions $f \in C_b([s,\infty)\times\mathbb{R}^d)$ such that $\frac{\partial}{\partial t}f$, $\frac{\partial}{\partial x_i}f$ and $\frac{\partial^2}{\partial x_i \partial x_j}f$ are elements of $C_b([s,\infty)\times\mathbb{R}^d)$ for all $i,j \in \{1,\dots,d\}$. Further, $C_{b,\infty}^{1,2}([s,\infty)\times\mathbb{R}^d)$ is the subspace of functions f for which $t \mapsto f(t,.)$ is continuous w.r.t. $\|.\|_\infty$. For every $s \geq 0$, $\eta \in \mathcal{M}_f(\mathbb{R}^d)$ and $f \in C_b^{1,2}([s,\infty)\times\mathbb{R}^d)$ we define

$$M_{s,t}(f) := \langle \bar{X}_t, f(t,.)\rangle - \langle \eta, f(s,.)\rangle - \int_s^t \langle \bar{X}_r, \frac{1}{2}\Delta f(r,.) + \frac{\partial}{\partial r}f(r,.)\rangle dr, \quad t \geq s.$$

Lemma 9.16 *For every $s \geq 0$, $\eta \in \mathcal{M}_f(\mathbb{R}^d)$ and $f \in C_b^{1,2}([s,\infty)\times\mathbb{R}^d)$, the process $(M_{s,t}(f) : t \geq s)$ is an $(\mathcal{F}_{[s,t]}^{\bar{X}})$-martingale under $\mathbb{P}_{s,\eta}$ and $\mathbb{E}_{s,\eta}[M_{s,t}^2(f)]$ is finite for $t \geq s$.*

Proof Using the first moment formula in (9.23) and the Markov property of $\bar{X}$ we obtain for every $t \geq s$ and $h \geq 0$:

$$\begin{aligned}
&\mathbb{E}_{s,\eta}\Big[M_{s,t+h}(f) - M_{s,t}(f)\Big|\mathcal{F}_{[s,t]}^{\bar{X}}\Big] \\
&= \mathbb{E}_{s,\eta}\Big[\langle \bar{X}_{t+h}, f(t+h,.)\rangle - \langle \bar{X}_t, f(t,.)\rangle - \int_t^{t+h} \langle \bar{X}_r, \frac{1}{2}\Delta f(r,.) + \frac{\partial}{\partial r}f(r,.)\rangle dr \Big|\mathcal{F}_{[s,t]}^{\bar{X}}\Big] \\
&= \mathbb{E}_{t,\bar{X}_t}\Big[\langle \bar{X}_{t+h}, f(t+h,.)\rangle\Big] - \langle \bar{X}_t, f(t,.)\rangle \\
&\qquad\qquad - \int_t^{t+h} \mathbb{E}_{t,\bar{X}_t}\Big[\langle \bar{X}_r, \frac{1}{2}\Delta f(r,.) + \frac{\partial}{\partial r}f(r,.)\rangle\Big] dr \\
&= \Big\langle \bar{X}_t\,,\, P_{(t+h)-t}f(t+h,.) - f(t,.) - \int_t^{t+h} P_{r-t}\Big(\frac{1}{2}\Delta f(r,.) + \frac{\partial}{\partial r}f(r,.)\Big)dr\Big\rangle \\
&= \Big\langle \bar{X}_t\,,\, P_h f(t+h,.) - f(t,.) - \int_0^{h} P_r\Big(\frac{1}{2}\Delta f(t+r,.) + \frac{\partial}{\partial v}f(v,.)|_{v=t+r}\Big)dr\Big\rangle \\
&= \int_{\mathbb{R}^d} \Big[(\tilde{P}_h f)(t,x) - f(t,x) - \int_0^h \tilde{P}_r\Big(\frac{1}{2}\Delta f + \frac{\partial}{\partial t}f\Big)(t,x)\,dr\Big]\,\bar{X}_t(dx)
\end{aligned}$$

where $(\tilde{P}_h f)(t,x) := (P_h f(t+h,.))(x)$. The operator family $(\tilde{P}_h)_{h\geq 0}$ provides a semigroup of linear bounded operators on $C_b^{1,2}([0,\infty)\times\mathbb{R}^d)$. Its infinitesimal generator (w.r.t. pointwise convergence) is $\tilde{L} = \frac{1}{2}\Delta + \frac{\partial}{\partial t}$. So we infer by means of a pointwise version of (3.21) that the integrand of the integral on the very r.h.s., and so the integral itself, vanishes. Hence, $M_{s,.}(f)$ is an $(\mathcal{F}_{[s,t]}^{\bar{X}})$-martingale under $\mathbb{P}_{s,\eta}$. The finiteness of the second moments can easily be shown with help of the second moment formula in (9.23). $\square$

Remark 9.17 *If we choose $f(t,x) = P_{T-t}\psi(x)$ (for some $T > s$ and $\psi \in C_b^2(\mathbb{R}^d)$) in the setting of Lemma 9.16, then the $[(\mathcal{F}_{[s,t]}^{\bar{X}}), \mathbb{P}_{s,\eta}]$-martingale turns into*

$$M_{s,t}(f) = \langle \bar{X}_t, P_{T-t}\psi(.)\rangle - \langle \eta, P_{T-s}\psi(.)\rangle, \qquad t \in [s,T]$$

since $\frac{1}{2}\Delta$ is the generator of the strongly continuous semigroup (P_t) on $(C_b^2, \|.\|_\infty^2)$.

From now on we assume that $\bar{X}$ is defined as a canonical *continuous* Markov process which is in particular a right Markov process; cf. the discussion at the end of Section 9.5. Hence, by Proposition 3.39, we may work with the usual augmentation $(\bar{\mathcal{F}}_{[s,t]}^{\bar{X},\mathbb{P}_{s,\eta}})$ of $(\mathcal{F}_{[s,t]}^{\bar{X}})$ w.r.t. $\mathbb{P}_{s,\eta}$. Since the usual augmentation satisfies the usual conditions, we do so. The continuity of $\bar{X}$ and Proposition 3.23 guarantee the following improvement of Lemma 9.16:

Lemma 9.18 *Assume that the catalytic SBM $\bar{X}$ is defined to be a canonical continuous Markov process. Then, for each $s \geq 0$, $\eta \in \mathcal{M}_f(\mathbb{R}^d)$ and $f \in C_{b,\infty}^{1,2}([s,\infty)\times\mathbb{R}^d)$, the process $(M_{s,t}(f) : t \geq s)$ is a continuous square-integrable $(\bar{\mathcal{F}}_{[s,t]}^{\bar{X},\mathbb{P}_{s,\eta}})$-martingale under $\mathbb{P}_{s,\eta}$.*

We next specify the quadratic variation process of $M_{s,\cdot}(f)$. Recall that $C_{[\bar{X},\varrho]}$ denotes the collision measure of $\bar{X}$ and its catalyst $\varrho(dtdx)$ (cf. Definition 9.13 and Theorem 9.14).

Lemma 9.19 *Assume that the catalytic SBM $\bar{X}$ is defined to be a canonical continuous Markov process. Then, for $s \geq 0$, $\eta \in \mathcal{M}_f(\mathbb{R}^d)$ and $f \in C_{b,\infty}^{1,2}([s,\infty) \times \mathbb{R}^d)$, the quadratic variation process of the $[(\bar{\mathcal{F}}_{[s,t]}^{\bar{X},\mathbb{P}_{s,\eta}}), \mathbb{P}_{s,\eta}]$-martingale $M_{s,\cdot}(f)$ is given by*

$$\langle M_{s,\cdot}(f)\rangle_t = \int_s^t \int_{\mathbb{R}^d} f^2(r,y) C_{[\bar{X},\varrho]}(drdy), \qquad t \geq s.$$

Proof First we note that the following formula holds for all $0 \leq s \leq t$, $\eta \in \mathcal{M}_f(\mathbb{R}^d)$ and $f \in C_{b,\infty}^{1,2}([s,\infty) \times \mathbb{R}^d)$:

$$\mathbb{E}_{s,\eta}\left[M_{s,t}^2(f)\right] = \int_{\mathbb{R}^d} \int_s^t \int_{\mathbb{R}^d} p_{r-s}(x,y) f^2(r,y) \varrho(drdy)\eta(dx). \tag{9.42}$$

Formula (9.42) is an immediate consequence of the moment formula (9.32) in Corollary 9.8 where $J_{t\vee t}(\tilde{s},x)$

$$= \begin{cases} (\tilde{P}_{t-\tilde{s}}f)(t,x) - (\tilde{P}_{s-\tilde{s}}f)(\tilde{s},x) - \int_{\tilde{s}}^t (\tilde{P}_{r-\tilde{s}}(\frac{1}{2}\Delta f + \frac{\partial}{\partial t}f))(\tilde{s},x)\, dr & , \quad \tilde{s} = s \\ (\tilde{P}_{t-\tilde{s}}f)(t,x) - \int_{\tilde{s}}^t (\tilde{P}_{r-\tilde{s}}(\frac{1}{2}\Delta f + \frac{\partial}{\partial t}f))(\tilde{s},x)\, dr & , \quad \tilde{s} \in (s,t] \\ 0 & , \quad \tilde{s} > t \end{cases}$$

$$= \begin{cases} 0 & , \quad \tilde{s} = s \\ f^2(\tilde{s},x) & , \quad \tilde{s} \in (s,t] \\ 0 & , \quad \tilde{s} > t \end{cases}$$

($(\tilde{P}_h)_{h\geq 0}$ is defined as in the proof of Lemma 9.16). Using the fact that $t \mapsto M_{s,t}(f)$ is an $(\bar{\mathcal{F}}_{[s,t]}^{\bar{X},\mathbb{P}_{s,\eta}})$-martingale, the Markov property of $\bar{X}$ and formula (9.42), we obtain for all $t \geq s$ and $h \geq 0$:

$$\begin{aligned} \mathbb{E}_{s,\eta}\Big[M_{s,t+h}^2(f)\Big|\bar{\mathcal{F}}_{[s,t]}^{\bar{X},\mathbb{P}_{s,x}}\Big] &= \mathbb{E}_{s,\eta}\Big[\Big(M_{s,t} + [M_{s,t+h}(f) - M_{s,t}(f)]\Big)^2\Big|\bar{\mathcal{F}}_{[s,t]}^{\bar{X},\mathbb{P}_{s,\eta}}\Big] \qquad (9.43) \\ &= M_{s,t}^2(f) + 2\times 0 + \mathbb{E}_{s,\eta}\Big[[M_{s,t+h}(f) - M_{s,t}(f)]^2\Big|\bar{\mathcal{F}}_{[s,t]}^{\bar{X},\mathbb{P}_{s,\eta}}\Big] \\ &= M_{s,t}^2(f) + \mathbb{E}_{t,\bar{X}_t}\Big[M_{t,t+h}(f)^2\Big] \\ &= M_{s,t}^2(f) + \int_{\mathbb{R}^d}\int_t^{t+h}\int_{\mathbb{R}^d} p_{r-t}(x,y) f^2(r,y)\varrho(drdy)\bar{X}_t(dx) \end{aligned}$$

$\mathbb{P}_{s,\eta}$-almost surely. By the Doob-Meyer decomposition (Theorem 3.22) we also have for some mean zero $(\bar{\mathcal{F}}_{[s,t]}^{\bar{X},\mathbb{P}_{s,\eta}})$-martingale $N_{s,.} = (N_{s,t} : t \geq s)$:

$$\begin{aligned}
&\mathbb{E}_{s,\eta}\Big[M_{s,t+h}^2(f) - M_{s,t}^2(f)\Big|\bar{\mathcal{F}}_{[s,t]}^{\bar{X},\mathbb{P}_{s,\eta}}\Big] \qquad (9.44)\\
&= \mathbb{E}_{s,\eta}\Big[N_{s,t+h} - N_{s,t}\Big|\bar{\mathcal{F}}_{[s,t]}^{\bar{X},\mathbb{P}_{s,\eta}}\Big] + \mathbb{E}_{s,\eta}\Big[\langle M_{s,.}(f)\rangle_{t+h} - \langle M_{s,.}(f)\rangle_t\Big|\bar{\mathcal{F}}_{[s,t]}^{\bar{X},\mathbb{P}_{s,\eta}}\Big].
\end{aligned}$$

From (9.43) and (9.44) we deduce

$$\mathbb{E}_{s,\eta}\Big[\langle M_{s,.}(f)\rangle_{t+h} - \langle M_{s,.}(f)\rangle_t\Big|\bar{\mathcal{F}}_{[s,t]}^{\bar{X},\mathbb{P}_{s,\eta}}\Big] = \int_{\mathbb{R}^d}\int_t^{t+h}\int_{\mathbb{R}^d} p_{r-t}(x,y) f^2(r,y)\varrho(drdy)\bar{X}_t(dx). \qquad (9.45)$$

In order to prove the claim of the lemma it is enough to show that the process

$$m_{s,t}(f) := \langle M_{s,.}(f)\rangle_t - \int_s^t\int_{\mathbb{R}^d} f^2(r,y) C_{[\bar{X},\varrho]}(drdy), \qquad t \geq s$$

(which is obviously of bounded variation and continuous) is a martingale. By means of $\bar{X}$'s Markov property, equation (9.45) and moment formula (9.37) we obtain

$$\begin{aligned}
&\mathbb{E}_{s,\eta}[m_{s,t+h}(f) - m_{s,t}(f)|\bar{\mathcal{F}}_{[s,t]}^{\bar{X},\mathbb{P}_{s,\eta}}]\\
&= \mathbb{E}_{s,\eta}\Big[\langle M_{s,.}(f)\rangle_{t+h} - \langle M_{s,.}(f)\rangle_t - \int_t^{t+h}\int_{\mathbb{R}^d} f^2(r,y)C_{[\bar{X},\varrho]}(drdy)\Big|\bar{\mathcal{F}}_{[s,t]}^{\bar{X},\mathbb{P}_{s,\eta}}\Big]\\
&= \mathbb{E}_{s,\eta}\Big[\langle M_{s,.}(f)\rangle_{t+h} - \langle M_{s,.}(f)\rangle_t\Big|\bar{\mathcal{F}}_{[s,t]}^{\bar{X},\mathbb{P}_{s,\eta}}\Big] - \mathbb{E}_{t,\bar{X}_t}\Big[\int_t^{t+h}\int_{\mathbb{R}^d} f^2(r,y)C_{[\bar{X},\varrho]}(drdy)\Big]\\
&= 0 \qquad \mathbb{P}_{s,\eta}\text{-almost surely.}
\end{aligned}$$

Hence $m_{s,.}(f)$ is indeed an $[(\bar{\mathcal{F}}_{[s,t]}^{\bar{X},\mathbb{P}_{s,\eta}}), \mathbb{P}_{s,\eta}]$-martingale. □

Lemmas 9.18 and 9.19 imply that $\bar{X}$ solves the following martingale problem:

Definition 9.20 [MARTINGALE PROBLEM] *An $\mathcal{M}_f(\mathbb{R}^d)$-valued continuous process $\bar{Y} = (\bar{Y}_t(dx) : t \geq s)$ with initial state $\bar{Y}_s = \eta \in \mathcal{M}_f(\mathbb{R}^d)$ on some probability space $[\Omega, \mathcal{F}, \mathbb{P}]$ is said to be a solution to the martingale problem $MP_{s,\eta}$ if the collision measure $C_{[\bar{Y},\varrho]}$ of $\bar{Y}$ and ϱ (in the sense of Definition 9.13) exists and if for every $f \in C_{b,\infty}^{1,2}([s,\infty)\times\mathbb{R})$:*

$$M_{s,t}(f) := \langle\bar{Y}_t, f(t,.)\rangle - \langle\eta, f(s,.)\rangle - \int_s^t\langle\bar{Y}_r, \frac{1}{2}\Delta f(r,.) + \frac{\partial}{\partial r}f(r,.)\rangle dr, \quad t \geq s$$

is a square-integrable continuous $(\bar{\mathcal{F}}_{[s,t]}^{\bar{Y},\mathbb{P}})$-martingale with quadratic variation process

$$\langle M_{s,.}(f)\rangle_t = \int_s^t\int_{\mathbb{R}^d} f^2(r,y)C_{[\bar{Y},\varrho]}(drdy), \qquad t \geq s.$$

The solution is said to be unique if any two solutions coincide in law.

The catalytic SBM is not only a solution to the martingale problem but even the only one:

Theorem 9.21 [UNIQUE SOLUTION TO MARTINGALE PROBLEM] *Let $\varrho(dtdx)$ satisfy condition* (B) *and $\bar{X} = [\bar{X}, \mathbb{P}_{s,\eta} : s \geq 0, \eta \in \mathcal{M}_f(\mathbb{R}^d)]$ be the corresponding canonical continuous catalytic SBM. Then, for every $s \geq 0$ and $\eta \in \mathcal{M}_f(\mathbb{R}^d)$, the process $(\bar{X}_t : t \geq s)$ under $\mathbb{P}_{s,\eta}$ is the unique solution to the martingale problem $MP_{s,\eta}$ posed in Definition 9.20.*

It remains to show the uniqueness claim of Theorem 9.21. Before proving that claim (the key is a duality argument) we present a useful corollary. For a similar result see [MRC88].

Corollary 9.22 [GREEN'S FUNCTION REPRESENTATION] *Suppose the assumptions of Theorem 9.21 are fulfilled. Then there exists, for every $s \geq 0$ and $\eta \in \mathcal{M}_f(\mathbb{R}^d)$, a continuous orthogonal martingale measure, M, with quadratic variation measure $\langle M \rangle(dtdx) = C_{[\bar{X},\varrho]}(dtdx)$ such that for every $t \geq s$ and $\psi \in C_b^2(\mathbb{R}^d)$:*

$$\langle \bar{X}_t, \psi \rangle = \langle \eta, P_{t-s}\psi \rangle + \int_s^t \int_{\mathbb{R}} P_{t-r}\psi(y) M(drdy) \quad \mathbb{P}_{s,\eta}\text{-almost surely.} \tag{9.46}$$

Proof For $\psi \in C_b^2(\mathbb{R}^d)$ set $f(t,x) := \psi(x)$ and let $M_{s,\cdot}(\psi) := M_{s,\cdot}(f)$ denote the continuous square-integrable martingale from Theorem 9.21. The class $(M_{s,\cdot}(\psi) : \psi \in C_b^2(\mathbb{R}^d))$ extends to a continuous orthogonal martingale measure $M = (M_t(A) : t \geq s, A \in \mathcal{A}(\mathbb{R}^d))$ with quadratic variation measure $\langle M \rangle(dtdx) = C_{[\bar{X},\varrho]}(dtdx)$ (proceed as in the proof of Lemma 6.2). In particular, we obtain for every $f \in C_{b,\infty}^{1,2}([s,\infty) \times \mathbb{R})$:

$$M_{s,t}(f) = \int_s^t \int_{\mathbb{R}^d} f(r,y) M(drdy) \qquad \forall t \geq s, \quad \mathbb{P}_{s,\eta}\text{-almost surely.} \tag{9.47}$$

The function f is admissible w.r.t. M since

$$\mathbb{E}_{s,\eta}\Big[\int_s^t \int_{\mathbb{R}^d} f^2(r,y) \langle M \rangle(drdy)\Big] = \mathbb{E}_{s,\eta}[\langle M_{s,\cdot}(f) \rangle_t] = \mathbb{E}_{s,\eta}[M_{s,t}^2(f)]$$

is finite for every $t \geq s$ by the square-integrability of $M_{s,\cdot}(f)$. In order to prove (9.46) fix $t \geq s$, $\psi \in C_b^2(\mathbb{R}^d)$ and set $f(r,y) = P_{t-r}\psi(y)$ for all $r \in [s,t]$. Then f belongs to $C_{b,\infty}^{1,2}([s,t] \times \mathbb{R}^d)$ and we deduce from (9.47) and Theorem 9.21 that

$$\langle \bar{X}_t, \psi \rangle = \langle \eta, P_{t-s}\psi \rangle + \int_s^t \langle \bar{X}_r, \frac{1}{2}\Delta P_{t-r}\psi + \frac{\partial}{\partial r} P_{t-r}\psi \rangle dr + \int_s^t \int_{\mathbb{R}^d} P_{t-r}\psi(y) M(drdy)$$

holds $\mathbb{P}_{s,\eta}$-almost surely. But the second summand on the r.h.s. vanishes since $\frac{1}{2}\Delta P_v \psi(x) = \frac{\partial}{\partial v} P_v \psi(x)$ holds for all $x \in \mathbb{R}^d$ and $v > 0$. This completes the proof. □

We now prove (the uniqueness claim in) Theorem 9.21. As mentioned at the beginning of Section 9.2, for every $\psi \in C_b^+(\mathbb{R}^d)$ the integral equation (9.8) has a unique non-negative, bounded and jointly continuous solution $(U_{s,t}\psi(x) : s \in [0,t], x \in \mathbb{R}^d)$. The same is true for equation (9.8) with $\varrho(drdy)$ replaced by the smoothed catalyst

$$\varrho^\epsilon(drdy) := \varrho^\epsilon(r,y)drdy := \int_{-\infty}^\infty \int_{\mathbb{R}^d} p_\epsilon(r,u) p_\epsilon(y,a) \varrho(duda)\, drdy$$

(here and later on we use the convention $\varrho(drdy) \equiv 0$ on $(-\infty, 0) \times \mathbb{R}^d$). We denote the solution by $(U^\epsilon_{s,t}\psi(x) : s \in [0,t] \times \mathbb{R}^d)$. This solution is not only jointly continuous but it is even smooth as the next lemma shows. Let $C_b^\infty([0,t] \times \mathbb{R}^d)$ denote the space of functions $f \in C_b([0,t] \times \mathbb{R}^d)$ such that $\frac{\partial^{m_0+m_1+\cdots+m_d}}{\partial t^{m_0}\partial x_1^{m_1}\cdots\partial x_d^{m_d}} f \in C_b([0,t] \times \mathbb{R}^d)$ for all $m_0, \ldots, m_d \geq 0$. The analogous space of continuous functions on $\mathbb{R}^d$ is denoted by $C_b^\infty(\mathbb{R}^d)$.

Lemma 9.23 *For every $t > 0$, $\epsilon > 0$ and $\psi \in C^\infty_{b,+}(\mathbb{R}^d)$, $(U^\epsilon_{s,t}\psi(x) : s \in [0,t], x \in \mathbb{R}^d)$ is in $C^\infty_{b,+}([0,t] \times \mathbb{R}^d)$ and solves the following BPDE rigorously:*

$$\begin{aligned} -\frac{\partial}{\partial s} u(s,t,x) &= \frac{1}{2}\Delta u(s,t,x) - \frac{1}{2}u^2(s,t,x)\varrho^\epsilon(s,x) \\ u(t,t,x) &= \psi(x) \qquad s \in [0,t],\ x \in \mathbb{R}^d. \end{aligned} \tag{9.48}$$

Proof $(\varrho^\epsilon(s,x) : s \in [0,t], x \in \mathbb{R}^d)$ is in $C^\infty_{b,+}([0,t] \times \mathbb{R}^d)$ since $\varrho(dtdx)$ is globally bounded (i.e. on balls with radius greater than 1) by a multiple of the Lebesgue measure, at least on $[0,T] \times \mathbb{R}^d$ for every $T > 0$. Proposition 3.2 of [Tay96] thus gives a unique solution to BPDE (9.48) in $C^\infty_{b,+}([0,t] \times \mathbb{R}^d)$. But standard arguments (cf. appendix of [Isc86]) show that this solution is also the unique solution $(U^\epsilon_{s,t}\psi(x) : s \in [0,t], x \in \mathbb{R}^d)$ to equation (9.8) with ϱ replaced by ϱ^ϵ. □

We next show that $U_{.,.}\psi(.)$ can be approximated by $U^\epsilon_{.,.}\psi(.)$ as $\epsilon \downarrow 0$.

Lemma 9.24 *For every $t > 0$ and Lipschitz continuous $\psi \in C_b^+(\mathbb{R}^d)$, $(U^\epsilon_{s,t}\psi(x) : s \in [0,t], x \in \mathbb{R}^d)$ converges to $(U_{s,t}\psi(x) : s \in [0,t], x \in \mathbb{R}^d)$ in $(C_b^+([0,t] \times \mathbb{R}^d), \|.\|_\infty)$ as $\epsilon \downarrow 0$.*

Proof Note that a similar result was proved in [DF92] (Theorem 2.13). However, in [DF92] the authors considered only a spatial smoothing of the catalyst. Here we need to generalize the arguments since we smoothed the catalyst in time as well. Fix $t > 0$ and a Lipschitz continuous $\psi \in C_b^+(\mathbb{R})$. Assume $\varrho(dtdx)$ satisfies condition (B) with β_1, β_2 and set $\beta := \beta_1/2 + \beta_2 - d/2$. Using Lemma 4.2$(i) \Rightarrow (ii)$ and Lemma 4.4(ii) we can find a finite constant $c'_t > 0$ such that

$$\sup_{\epsilon \in (0,1]} \sup_{x \in \mathbb{R}^d} \int_s^{s'} \int_{\mathbb{R}^d} p_{r-s}(x,y)\varrho^\epsilon(drdy) \ \leq\ c'_t\,|s - s'|^\beta \qquad \forall s, s' \in [0,t] : s \leq s'. \tag{9.49}$$

Also, $\|2(P_{t-s}\psi)^2\|_\infty \leq c''_{t,\psi}$ holds for all $s \in [0,t]$ for some $c''_{t,\psi} > 0$. Then pick $\delta > 0$ small enough such that such that $q := c'_t c''_{t,\psi}\delta^\beta \in (0,1)$. Divide the interval $[0,t]$ in pieces of length δ, i.e. consider the partition $[0,t] = (\cup_{k=1}^K [t - k\delta, t - (k-1)\delta]) \cup [0, t - K\delta]$ where K is the largest integer with $K\delta < t$. Set $M^\epsilon_{m,n} := \sup_{r \in [t-n\delta, t-m\delta]} \|U_{r,t}\psi(.) - U^\epsilon_{r,t}\psi(.)\|_\infty$ and $M^\epsilon_{n,K+1} := \sup_{r \in [0,t-n\delta]} \|U_{r,t}\psi(.) - U^\epsilon_{r,t}\psi(.)\|_\infty$ for integers m, n with $m < n \leq K$. Suppose there is a $k \in \{1, \ldots, K\}$ such that $\lim_{\epsilon \downarrow 0} M^\epsilon_{0,k} = 0$. Let $\theta > 0$. Then we have for all $s \in [t - (k+1)\delta, t - k\delta]$, $x \in \mathbb{R}^d$ and $\epsilon \in (0,1]$:

$$|U_{s,t}\psi(x) - U^\epsilon_{s,t}\psi(x)|$$

$$
\begin{aligned}
&\le \int_s^{t-k\delta}\int_{\mathbb{R}^d} p_{r-s}(x,y)\left|(U_{r,t}\psi)^2(y)-(U^\epsilon_{r,t}\psi)^2(y)\right|\varrho^\epsilon(drdy)\\
&\quad+\int_{t-k\delta}^t\int_{\mathbb{R}^d} p_{r-s}(x,y)\left|(U_{r,t}\psi)^2(y)-(U^\epsilon_{r,t}\psi)^2(y)\right|\varrho^\epsilon(drdy)\\
&\quad+\int_s^{(s+\theta)\wedge t}\int_{\mathbb{R}^d} p_{r-s}(x,y)(U_{r,t}\psi)^2(y)\Big(\varrho(drdy)+\varrho^\epsilon(drdy)\Big)\\
&\quad+\Big|\int_{(s+\theta)\wedge t}^t\int_{\mathbb{R}^d} p_{r-s}(x,y)(U_{r,t}\psi)^2(y)\Big(\varrho(drdy)-\varrho\epsilon(drdy)\Big)\Big|\\
&\le qM^\epsilon_{k,k+1}+c_{t,\psi}M^\epsilon_{0,k}+c_{t,\psi}\theta^\beta\\
&\quad+\Big|\int_{(s+\theta)\wedge t}^t\int_{\mathbb{R}^d} p_{r-s}(x,y)(U_{r,t}\psi)^2(y)\Big(\varrho(drdy)-\varrho^\epsilon(drdy)\Big)\Big|. \qquad (9.50)
\end{aligned}
$$

Here we used the domination of $U_{s,t}\psi(.)$ and $U^\epsilon_{s,t}\psi(.)$ by $P_{t-s}\psi(.)$, the elementary estimate $|a^2-b^2|\le(a+b)|a-b|$ for $a,b\in\mathbb{R}_+$ and (9.49). Suppose the integral on the r.h.s. of (9.50), henceforth denoted by $I^\epsilon(s,x)$, converges to 0 as $\epsilon\downarrow 0$ uniformly in $s\in[t-(k+1)\delta,t-k\delta]$ and $x\in\mathbb{R}^d$ for every $\theta>0$. Then we have

$$
0\le\limsup_{\epsilon\downarrow 0}M^\epsilon_{k,k+1}\le q\limsup_{\epsilon\downarrow 0}M^\epsilon_{k,k+1}+c_{t,\psi}\theta^\beta \qquad \forall\theta>0
$$

which implies $\lim_{\epsilon\downarrow 0}M^\epsilon_{k,k+1}=0$ since $q\in(0,1)$. Thus, $\lim_{\epsilon\downarrow 0}M^\epsilon_{0,k+1}=0$ holds and the claim of Lemma 9.24 follows by induction.

It remains to show $\lim_{\epsilon\downarrow 0}I^\epsilon(s,x)=0$ uniformly in s and x for every $\theta>0$. Define $f(r,y|s,x)=f(r,y|s,\theta,t,x,\psi):=p_{r-s}(x,y)(U_{r,t}\psi)^2(y)$ for $(r,y)\in[(s+\theta)\wedge t,t]\times\mathbb{R}^d$ and extend $f(.,.|s,x)$ continuously to $(-\infty,\infty)\times\mathbb{R}^d$ in such a way that $(f(r,.|s,x):r\in(-\infty,\infty))$ is constant outside $[(s+\theta)\wedge t,t]$. In particular, $\sup_{s\in[0,t],x\in\mathbb{R}^d}\|f(.,.|s,x)\|_\infty<\infty$. Then, using the elementary splitting $\int_{\mathbb{R}^{1+d}}\int_A=\int_A\int_{\mathbb{R}^{1+d}}-\int_A\int_{A^c}+\int_{A^c}\int_A$ with $A=[(s+\theta)\wedge t,t]\times\mathbb{R}^d$, we obtain

$$
\begin{aligned}
I^\epsilon(s,x) &= \Big|\int_{(s+\theta)\wedge t}^t\int_{\mathbb{R}^d}\int_{-\infty}^\infty\int_{\mathbb{R}^d} p_\epsilon(r,v)p_\epsilon(y,b)p_{r-s}(x,y)(U_{r,t}\psi)^2(y)dbdv\varrho(drdy)\\
&\quad-\int_{-\infty}^\infty\int_{\mathbb{R}^d}\int_{(s+\theta)\wedge t}^t\int_{\mathbb{R}^d} p_\epsilon(r,u)p_\epsilon(y,a)p_{r-s}(x,y)(U_{r,t}\psi)^2(y)dydr\varrho(duda)\Big|\\
&\le \int_{(s+\theta)\wedge t}^t\int_{\mathbb{R}^d}\int_{-\infty}^\infty\int_{\mathbb{R}^d} p_\epsilon(r,v)p_\epsilon(y,b)\Big|f(r,y|s,x)-f(v,b|s,x)\Big|dbdv\varrho(drdy)\\
&\quad+\int_{(s+\theta)\wedge t}^t\int_{\mathbb{R}^d}\int_{[(s+\theta)\wedge t,t]^c}\int_{\mathbb{R}^d} p_\epsilon(r,u)p_\epsilon(y,a)f(r,y|s,x)dydr\varrho(duda)\\
&\quad+\int_{[(s+\theta)\wedge t,t]^c}\int_{\mathbb{R}^d}\int_{(s+\theta)\wedge t}^t\int_{\mathbb{R}^d} p_\epsilon(r,u)p_\epsilon(y,a)f(r,y|s,x)dydr\varrho(duda)\\
&=: I^\epsilon_1(s,x)+I^\epsilon_2(s,x)+I^\epsilon_3(s,x).
\end{aligned}
$$

We first show that $I^\epsilon_2(s,x)$ and $I^\epsilon_3(s,x)$ vanish as $\epsilon\downarrow 0$. Let $\gamma>0$. $I^\epsilon_2(s,x)$ is bounded by

$$
\int_{(s+\theta)\wedge t}^t\int_{\mathbb{R}^d}\int_{[(s+\theta)\wedge t,t]^c}\int_{\mathbb{R}^d} p_\epsilon(r,u)p_\epsilon(y,a)\,c_{t,\psi,\theta}\,e^{-\frac{|x-y|^2}{2t}}dydr\varrho(duda)
$$

$$
\begin{aligned}
&\leq c_{t,\psi,\theta} \int_{(s+\theta)\wedge t}^{t} \int_{[(s+\theta)\wedge t,t]^c} p_\epsilon(r,u) \int_{\mathbb{R}^d} (P_\epsilon e^{-\frac{|x-.|^2}{2t}})(a)\varrho_1(u,da)dr\varrho_2(du) \\
&\leq c'_{t,\psi,\theta} \int_{(s+\theta)\wedge t}^{t} \int_{[(s+\theta)\wedge t,t]^c} p_\epsilon(r,u)dr\varrho_2(du) \\
&= c'_{t,\psi,\theta}\Big(\int_{((s+\theta)\wedge t)+\gamma}^{t-\gamma} \int_{[(s+\theta)\wedge t,t]^c} p_\epsilon(r,u)dr\varrho_2(du) \\
&\quad + \int_{(s+\theta)\wedge t}^{((s+\theta)\wedge t)+\gamma} \int_{[(s+\theta)\wedge t,t]^c} p_\epsilon(r,u)dr\varrho_2(du) + \int_{t-\gamma}^{t} \int_{[(s+\theta)\wedge t,t]^c} p_\epsilon(r,u)dr\varrho_2(du)\Big) \\
&\leq c'_{t,\psi,\theta}\Big(\int_{((s+\theta)\wedge t)+\gamma}^{t-\gamma} \Big\{ \int_{B(u,\gamma)^c} p_\epsilon(r,u)dr\Big\}\varrho_2(du) + \gamma^{\beta_2} + \gamma^{\beta_2}\Big)
\end{aligned}
$$

for every $\epsilon \in (0,1]$, $s \in [0,t]$ and $x \in \mathbb{R}^d$. Taking $\lim_{\epsilon\downarrow 0} \sup_{u\in\mathbb{R}} \int_{B(u,\gamma)^c} p_\epsilon(r,u)dr = 0$ into account we can find for every (sufficiently small) $\bar{\gamma} > 0$ an $\epsilon_{\bar{\gamma}} > 0$ such that $I_2^\epsilon(s,x) < \bar{\gamma}$ for all $\epsilon \in (0,\epsilon_{\bar{\gamma}}]$, $s \in [0,t]$ and $x \in \mathbb{R}^d$. Hence, $\lim_{\epsilon\downarrow 0} I_2^\epsilon(s,x) = 0$ uniformly in s and x. The convergence of I_3^ϵ can be shown analogously.

To show $\lim_{\epsilon\downarrow 0} I_1^\epsilon(s,x) = 0$ we need the following two estimates. On the one hand,

$$
|U_{r,t}\psi(y) - U_{u,t}\psi(b)| \leq c_{t,\psi}(|r-u|^\beta + |y-b|^{2\beta}) \tag{9.51}
$$

holds for all $r,u \in [0,t]$ and $y,b \in \mathbb{R}^d$. On the other hand, we have

$$
|p_{r-s}(x,y) - p_{u-s}(x,b)| \leq c_{t,\theta}\, (e^{-\frac{(|x-y|-1)^2}{4t}} + e^{-\frac{|x-y|^2}{4t}})(|r-u| + |y-b|) \tag{9.52}
$$

for all $s \in [0,t]$, $r,u \in [(s+\theta)\wedge t, t]$ and $x,y,b \in \mathbb{R}^d$ with $|y-b| \leq 1$. In order to prove inequality (9.51) we note that $|U_{r,t}\psi(y) - U_{u,t}\psi(b)|$ is bounded by

$$
|P_{t-r}\psi(y) - P_{t-u}\psi(b)| + \frac{1}{2}\int_{r\wedge u}\int_{\mathbb{R}^d} |p_{v-r}(a,y) - p_{v-u}(a,b)|\, \|(U_{r,t}\psi)^2(.)\|_\infty\, \varrho(dvda)
$$

(where $p_v :\equiv 0$ for $v < 0$). Then (9.51) can be deduced with help of the Lipschitz continuity of ψ and Lemma 4.9 as well as Lemma 4.7 and $\sup_{r\in[0,t]} \|(U_{r,t}\psi)^2(.)\|_\infty < \infty$. Inequality (9.52) is a consequence of Lemma 4.5(i) and (iii). From (9.51) and (9.52) we can immediately deduce

$$
|f(r,y|s,x) - f(u,b|s,x)| \leq c_{t,\theta,\psi}\, (e^{-\frac{(|y-z|-1)^2}{4t}} + e^{-\frac{(y-z)^2}{4t}})(|r-u|^\beta + |y-b|^{2\beta}) \tag{9.53}
$$

for all $s \in [0,t]$, $r,u \in [(s+\theta)\wedge t, t]$ and $x,y,b \in \mathbb{R}^d$ with $|y-b| \leq 1$. Let $\gamma \in (0,\theta/2)$. We split $I_1^\epsilon(s,x)$ into

$$
\begin{aligned}
&I_{1,1}^\epsilon(s,x) + I_{1,2}^\epsilon(s,x) := \\
&\quad \int_{(s+\theta)\wedge t}^{t} \int_{\mathbb{R}^d} \int_{B(r,\gamma)} \int_{B(y,\gamma)} p_\epsilon(r,v)p_\epsilon(y,b)\Big|f(r,y|s,x) - f(v,b|s,x)\Big|dbdv\varrho(drdy) + \\
&\quad \int_{(s+\theta)\wedge t}^{t} \int_{\mathbb{R}^d} \int_{B(r,\gamma)^c} \int_{B(y,\gamma)^c} p_\epsilon(r,v)p_\epsilon(y,b)\Big|f(r,y|s,x) - f(v,b|s,x)\Big|dbdv\varrho(drdy).
\end{aligned}
$$

By means of (9.53), $I_{1,1}^{\epsilon}(s,x)$ can be estimated by

$$\int_{(s+\theta)\wedge t}^{t}\int_{\mathbb{R}^d}\int_{B(r,\gamma)}\int_{B(y,\gamma)} p_\epsilon(r,v)p_\epsilon(y,b) \\ \times c_{t,\theta,\psi}(e^{-\frac{(|x-y|-1)^2}{4t}}+e^{-\frac{|x-y|^2}{4t}})(\gamma^\beta+\gamma^{2\beta})dbdv\varrho(drdy) \;\leq\; \bar{c}_{t,\theta,\psi}\gamma^\beta$$

for $\epsilon \in (0,1]$, $s \in [0,t]$ and $x \in \mathbb{R}^d$. Also, we can estimate $\int_{B(y,\gamma)^c} p_\epsilon(y,b)e^{-\frac{|x-b|^2}{2t}}db$ by

$$\Big(\int_{B(y,\gamma)^c} p_\epsilon(y,b)db\Big)^{1/2}\Big((2\pi(t/2))^{d/2}\, p_{t/2+\epsilon}(x,y)\Big)^{1/2}$$

for which we used Hölder's inequality. Using this we get the following bound for $I_{1,2}^{\epsilon}(s,x)$

$$\begin{aligned}
&\int_{(s+\theta)\wedge t}^{t}\int_{\mathbb{R}^d}\int_{B(r,\gamma)^c}\int_{B(y,\gamma)^c} p_\epsilon(r,v)p_\epsilon(y,b)c_{t,\theta}\Big(e^{-\frac{|x-y|^2}{2t}}+e^{-\frac{|x-b|^2}{2t}}\Big)dbdv\varrho(drdy)\\
\leq\;& c_{t,\theta}\int_{(s+\theta)\wedge t}^{t}\int_{\mathbb{R}^d}\int_{B(r,\gamma)^c} p_\epsilon(r,v)\int_{B(y,\gamma)^c} p_\epsilon(y,b)db\;dv\; e^{-\frac{|x-y|^2}{2t}}\varrho(drdy)+\\
& c'_{t,\theta}\int_{(s+\theta)\wedge t}^{t}\int_{\mathbb{R}^d}\int_{B(r,\gamma)^c} p_\epsilon(r,v)\Big(\int_{B(y,\gamma)^c} p_\epsilon(y,b)db\Big)^{1/2}dv\; e^{-\frac{|x-y|^2}{t+2}}\varrho(drdy)\\
\leq\;& c''_{t,\theta}\max_{i=1,2}\sup_{z\in\mathbb{R}^d}\Big(\int_{B(z,\gamma)^c} p_\epsilon(z,b)db\Big)^{i/2}\int_{(s+\theta)\wedge t}^{t}\int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{t+2}}\varrho(drdy)\\
\leq\;& c'''_{t,\theta}\max_{i=1,2}\sup_{z\in\mathbb{R}^d}\Big(\int_{B(z,\gamma)^c} p_\epsilon(z,b)db\Big)^{i/2}
\end{aligned}$$

for all $\epsilon \in (0,1]$, $s \in [0,t]$ and $x \in \mathbb{R}^d$. The latter estimate tends to 0 as $\epsilon \downarrow 0$ since $\gamma > 0$. Hence we can find for every (sufficiently small) $\bar{\gamma} > 0$ an $\epsilon_{\bar{\gamma}} > 0$ such that $I_1^{\epsilon}(s,x) < \bar{\gamma}$ for all $\epsilon \in (0,\epsilon_{\bar{\gamma}}]$, $s \in [0,t]$ and $x \in \mathbb{R}^d$. That is, $\lim_{\epsilon\downarrow 0} I_1^{\epsilon} = 0$ uniformly in s and x. On the whole we obtain $\lim_{\epsilon\downarrow 0} I^{\epsilon}(s,x) = 0$ uniformly in $s \in [t-(k+1)\delta, t-k\delta]$ and $x \in \mathbb{R}^d$, for every $\theta > 0$, which finishes the proof of Lemma 9.24. □

As the next step we show duality of any solution $[\bar{Y},\Omega,\mathcal{F},\mathbb{P}]$ of the martingale problem and $U_{\cdot,\cdot}\psi(.)$ w.r.t. the mapping $\mathcal{M}_f(\mathbb{R}^d)\times C_{b,+}^{\infty}(\mathbb{R}^d)\to\mathbb{R}$, $(\eta,\psi)\mapsto e^{-\langle\eta,\psi\rangle}$. That is,

Lemma 9.25 *Let $[\bar{Y},\mathbb{P}] = [(\bar{Y}_t(dx) : t \geq s),\Omega,\mathcal{F},\mathbb{P}]$ be any solution to the martingale problem $MP_{s,\eta}$ from Definition 9.20. Then we have for every $t \geq s$ and $\psi \in C_{b,+}^{\infty}(\mathbb{R}^d)$:*

$$\mathbb{E}\Big[e^{-\langle\bar{Y}_t,\psi\rangle}\Big] = e^{-\langle\eta,U_{s,t}\psi(.)\rangle}.$$

Proof As in the proof of Corollary 9.22 one can show the existence of a continuous orthogonal martingale measure, M, with quadratic variation measure $\langle M\rangle(dtdx) = C_{[\bar{Y},\varrho]}(dtdx)$ and satisfying

$$\langle\bar{Y}_r, f(r,.)\rangle = \langle\eta, f(s,.)\rangle + \int_s^r\langle\bar{Y}_u,\frac{\partial}{\partial u}f(u,.)+\frac{1}{2}\Delta f(u,.)\rangle du + \int_s^r\int_{\mathbb{R}^d} f(u,y)M(dudy)$$

for every $r \geq s$ and $f \in C_{b,\infty}^{1,2}([s,\infty) \times \mathbb{R}^d)$. Set $f(r,y) = U_{r,t}^\epsilon \psi(y)$ for all $r \in [s,t]$ and $y \in \mathbb{R}^d$. Since ψ is Lipschitz continuous (recall $\psi \in C_{b,+}^\infty(\mathbb{R}^d)$), one can show that $t \mapsto f(t,.)$ is $\|.\|_\infty$-continuous; in particular, $f \in C_{b,\infty}^{1,2}([s,t] \times \mathbb{R}^d)$. Then, Itô's formula (applied to the semimartingale $[s,t] \ni r \mapsto \langle \bar{Y}_r, f(r,.)\rangle$ and $x \mapsto e^{-x}$) and Lemma 9.23 yield

$$\begin{aligned}
N_{s,t}^\epsilon &:= -\int_s^t \int_{\mathbb{R}^d} e^{-\langle \bar{Y}_r, U_{r,t}^\epsilon \psi(.)\rangle} U_{r,t}^\epsilon \psi(.) M(drdy) \\
&= e^{-\langle \bar{Y}_t, \psi\rangle} - e^{-\langle \eta, U_{s,t}^\epsilon \psi(.)\rangle} \\
&\quad + \int_s^t e^{-\langle \bar{Y}_r, U_{r,t}^\epsilon \psi(.)\rangle} \Big\langle \bar{Y}_r, \frac{1}{2}\Delta U_{r,t}^\epsilon \psi(.) + \frac{\partial}{\partial r} U_{r,t}^\epsilon \psi(.) \Big\rangle dr \\
&\quad - \frac{1}{2}\int_s^t \int_{\mathbb{R}^d} e^{-\langle \bar{Y}_r, U_{r,t}^\epsilon \psi(.)\rangle} (U_{r,t}^\epsilon \psi)^2(y) \langle M\rangle(drdy) \\
&= \Big(e^{-\langle \bar{Y}_t, \psi\rangle} - e^{-\langle \eta, U_{s,t}^\epsilon \psi(.)\rangle}\Big) \\
&\quad + \Big(\int_s^t \int_{\mathbb{R}^d} e^{-\langle \bar{Y}_r, U_{r,t}^\epsilon \psi(.)\rangle} \frac{1}{2} (U_{r,t}^\epsilon \psi)^2(y) \varrho^\epsilon(r,y) \bar{Y}_r(dy) dr - \\
&\qquad \int_s^t \int_{\mathbb{R}^d} e^{-\langle \bar{Y}_r, U_{r,t}^\epsilon \psi(.)\rangle} \frac{1}{2} (U_{r,t}^\epsilon \psi)^2(y) C_{[\bar{Y},\varrho]}(drdy)\Big) \\
&=: I_1^\epsilon + I_2^\epsilon.
\end{aligned}$$

By Lemma 9.24, I_1^ϵ converges to $e^{-\langle \bar{Y}_t, \psi\rangle} - e^{-\langle \eta, U_{s,t}\psi(.)\rangle}$ as $\epsilon \downarrow 0$. Using the definition of $C_{[\bar{Y},\varrho]}(drdy)$, I_2^ϵ can easily be shown to converge $\mathbb{P}$-almost surely to 0 as $\epsilon \downarrow 0$. Also, by Lemma 3.5 the family $\{N_{s,t}^\epsilon : \epsilon > 0\}$ is uniformly integrable since

$$\begin{aligned}
\sup_{\epsilon>0} \mathbb{E}[(N_{s,t}^\epsilon)^2] &= \sup_{\epsilon>0} \mathbb{E}\Big[\int_s^t \int_{\mathbb{R}^d} e^{-2\langle \bar{Y}_r, U_{r,t}^\epsilon \psi(.)\rangle} (U_{r,t}^\epsilon \psi)^2(y) \langle M\rangle(drdy)\Big] \\
&\leq \mathbb{E}\Big[\int_s^t \int_{\mathbb{R}} \|P_{t-r}\psi\|_\infty^2 \langle M\rangle(drdy)\Big] \\
&\leq c_{t,\psi}\, \mathbb{E}[\langle M_{s,.}(\mathbf{1})\rangle_t] \;=\; c_{t,\psi}\, \mathbb{E}[M_{s,t}^2(\mathbf{1})]
\end{aligned}$$

is finite by the square-integrability of $M_{s,.}(\mathbf{1})$. The uniform integrability and the $\mathbb{P}$-almost sure convergence imply $L^1(\mathbb{P})$-convergence of $N_{s,t}^\epsilon$ to $e^{-\langle \bar{Y}_t, \psi\rangle} - e^{-\langle \eta, U_{s,t}\psi(.)\rangle}$ as $\epsilon \downarrow 0$ since $e^{-\langle \bar{Y}_t, \psi\rangle} - e^{-\langle \eta, U_{s,t}\psi(.)\rangle} \in L^1(\mathbb{P})$, cf. Proposition 3.12 of [Kal97]. In particular we obtain $\lim_{\epsilon\downarrow 0} \mathbb{E}[N_{s,t}^\epsilon] = \mathbb{E}[e^{-\langle \bar{Y}_t, \psi\rangle}] - e^{-\langle \eta, U_{s,t}\psi(.)\rangle}$. Then the claim follows from the fact that $\mathbb{E}[N_{s,t}^\epsilon] = 0$ for every $\epsilon > 0$. □

As an immediate consequence of Lemma 9.25 we obtain uniqueness of the one-dimensional distributions of solutions to $\mathrm{MP}_{s,\eta}$ via the Laplace transform (apply Corollary 3.32). But uniqueness of the one-dimensional distributions assures uniqueness in law as the following lemma shows. This completes the proof of Theorem 9.21; as already mentioned, Lemmas 9.18 and 9.19 imply that the catalytic SBM $[\bar{X}, \mathbb{P}_{s,\eta}]$ solves the martingale problem $\mathrm{MP}_{s,\eta}$.

Lemma 9.26 *Let $s \geq 0$ and $\eta \in \mathcal{M}_f(\mathbb{R}^d)$. Suppose that any two solutions to $MP_{s,\eta}$ have the same one-dimensional distributions. Then any two solutions to $MP_{s,\eta}$ have the same finite-dimensional distributions, i.e. they coincide in law.*

Proof The key is Mytnik's observation (cf. [Myt98], Section 2) that the proof of Theorem 4.4.2 of [EK86] also works for a richer class of martingale problems than the class considered there. In particular, the proof of Lemma 9.26 can be carried out completely analogously to the proof of Theorem 4.4.2 of [EK86]. □

9.8 Jointly continuous density field

We now turn to the main result of this chapter. We give a rather satisfying answer to the question: Under which assumptions on the catalyst $\varrho(dtdx)$ does the catalytic SBM $\bar{X}$ possess a jointly continuous Lebesgue density field? A discussion of this problem was already presented in the Introduction (Chapter 1). We mentioned there that the existence of a space-time regular density seems to be possible only in dimension $d = 1$ under some restriction on the local concentration of catalytic mass. Condition (A) from Definition 2.21 turns out to be a proper condition on the catalyst $\varrho(dtdx)$. On the one hand, any catalyst satisfying condition (A) induces a jointly continuous density field for the corresponding catalytic SBM. On the other hand, a slight violation may lead to discontinuities of the density (cf. (1.6)). Note that there is a large class of non-atomic catalysts which satisfy condition (A); examples were given in Section 2.8. We henceforth assume that the catalyst $\varrho(dtdx)$ satisfies condition (A). Hence we restrict in particular to $d = 1$. Recall that condition (A) is stronger than (B), i.e. the catalyst is admissible. By Theorem 9.11 we may and do assume the corresponding catalytic SBM to be weakly continuous.

Theorem 9.27 [SPACE-TIME REGULARITY] *Let $\varrho(dtdx)$ satisfy condition* (A) *with α_1, α_2 and let $\bar{X} = [\bar{X}, \mathbb{P}_{s,\eta} : s \geq 0, \eta \in \mathcal{M}_f(\mathbb{R})]$ denote the corresponding canonical continuous catalytic SBM. Pick $s \geq 0$ and $\eta \in \mathcal{M}_f(\mathbb{R})$. Then $\bar{X}_t(dx)$ has a Lebesgue density $X_t(.)$ for all times $t > s$ and $X = (X_t(x) : t > s, x \in \mathbb{R})$ can be chosen to be jointly continuous, $\mathbb{P}_{s,\eta}$-almost surely. Moreover, X is locally jointly Hölder-γ-continuous for all $\gamma \in (0, \alpha/2)$ where $\alpha := \alpha_1/2 + \alpha_2 - 1$.*

The remainder of this section is devoted to the proof of Theorem 9.27. We first prove a Lemma concerning the moments of $\bar{X}$.

Lemma 9.28 *Let $m \geq 1$ and $s < t_0 < T$. Then there exists a finite constant $c > 0$ depending only on m, t_0, T, η and ϱ such that for all $t \in [t_0, T]$, $x \in \mathbb{R}$ and $\epsilon \in (0, 1]$:*

$$\mathbb{E}_{s,\eta}\left[\langle \bar{X}_t, p_\epsilon(x, .)\rangle^m\right] \leq c.$$

Proof Fix $t \in [t_0, T]$ and $x \in \mathbb{R}$. By Theorem 9.5,

$$\mathbb{E}_{s,\eta}\left[\langle \bar{X}_t, p_\epsilon(x, .)\rangle^m\right] = m! \sum_{k=1}^{m} \frac{(-1)^{m+k}}{k!} \sum_{\substack{n_1,\dots,n_k \geq 1 \\ n_1+\cdots+n_k = m}} \prod_{i=1}^{k} \langle \eta, a_{n_i}(s, .|t, J_t)\rangle, \tag{9.54}$$

where the $a_n(., .|t, J_t)$ are defined as in Lemma 9.4 with $J_t(s, z) := p_{t-s+\epsilon}(x, z)$. We shall show that for each $n \geq 1$ there is a finite constant $C_n > 0$ depending only on T and ϱ

(apart from n) such that for all $s \in [0,t]$ and $z \in \mathbb{R}$:

$$|a_n(s,z)| \ \leq \ \frac{C_n}{\sqrt{t-s+\epsilon}}\, e^{-\frac{(x-z)^2}{2^n(t-s+\epsilon)}}. \tag{9.55}$$

Then the claim follows from (9.54). We proceed by induction on n. For $n = 1$ (9.55) trivially holds with $C_1 = 1/\sqrt{2\pi}$. For $n \geq 2$ we obtain by the assumption for $n-1$, Lemma 4.2$(i) \Rightarrow (ii)$, Lemma 4.4(i) and the inequality

$$\begin{aligned} &\exp(-\frac{(y-z)^2}{2(r-s)})\exp(-\frac{(x-y)^2}{2^{n-1}(t-r+\epsilon)}) \\ &\leq \ \exp(-\frac{(y-z)^2+(x-y)^2}{2^{n-1}(t-s+\epsilon)}) \ \leq \ \exp(-\frac{(x-z)^2}{2^n(t-s+\epsilon)}) \qquad \forall r \in [s,t] \end{aligned}$$

the following estimate:

$$\begin{aligned} |a_n(s,z)| \ &\leq \ \frac{1}{2}\int_s^\infty \int_{\mathbb{R}} p_{r-s}(y,z)\Big(\sum_{j=1}^{n-1} |a_j(r,y)a_{n-j}(r,y)|\Big)\varrho(drdy) \\ &\leq \ \frac{1}{2}\int_s^t \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi(r-s)}}\, e^{-\frac{(y-z)^2}{2(r-s)}} \\ &\qquad \times\Big(\sum_{j=1}^{n-1} \frac{C_j}{\sqrt{t-r+\epsilon}}\, e^{-\frac{(x-y)^2}{2^{n-1}(t-r+\epsilon)}}\, \frac{C_{n-j}}{\sqrt{t-r+\epsilon}}\, e^{-\frac{(x-y)^2}{2^{n-1}(t-r+\epsilon)}}\Big)\varrho(drdy) \\ &\leq \ C_n'\, e^{-\frac{(x-z)^2}{2^n(t-s+\epsilon)}} \int_s^t \frac{1}{\sqrt{r-s}}\, \frac{1}{t-r+\epsilon} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{2^{n-1}(t-r+\epsilon)}}\varrho_1(r,dy)\varrho_2(dr) \\ &\leq \ C_n'\, e^{-\frac{(x-z)^2}{2^n(t-s+\epsilon)}} \int_s^t \frac{1}{\sqrt{r-s}}\, \frac{1}{t-r+\epsilon}\, \bar{c}_{n,T}\, (2^{n-1}(t-r+\epsilon))^{\alpha_1/2}\varrho_2(dr) \\ &\leq \ C_{n,T}'\, e^{-\frac{(x-z)^2}{2^n(t-s+\epsilon)}}\Big(\frac{1}{(t-s+\epsilon)/2}\int_s^{s+(t-s+\epsilon)/2} \frac{1}{\sqrt{r-s}}\,\varrho_2(dr) \\ &\qquad + \frac{1}{\sqrt{(t-s+\epsilon)/2}}\int_{s+(t-s+\epsilon)/2}^t \frac{1}{(t-r+\epsilon)^{1-\alpha_1/2}}\,\varrho_2(dr)\Big) \\ &\leq \ C_{n,T}''\, e^{-\frac{(x-z)^2}{2^n(t-s+\epsilon)}}\, \frac{1}{(t-s+\epsilon)^{1/2-(\alpha_1/2+\alpha_2-1)}} \ \leq \ C_n\, e^{-\frac{(x-z)^2}{2^n(t-s+\epsilon)}}\, \frac{1}{\sqrt{t-s+\epsilon}} \end{aligned}$$

where $C_n' = \frac{1}{2}(2\pi)^{-1/2}(n-1)\max\{C_j^2 : j = 1,\ldots,n-1\}$. Hence (9.55) holds. $\square$

We now present the strategy of the proof of Theorem 9.27. Similar arguments have been used by Konno and Shiga ([KS88], Theorem 1.4) for the case of the classical SBM; see also Perkin's Lecture Notes ([Per02], Section III.4). Define a smoothed version X^ϵ of $\bar{X}$ by $X_t^\epsilon(x) := \langle \bar{X}_t, p_\epsilon(x,.)\rangle$, $t \geq s$ and $x \in \mathbb{R}$. As the first step (Step 1) we shall show that every X^ϵ is an element of the space $\mathcal{P}_s$ of $(\bar{\mathcal{F}}_{[s,t]}^{\bar{X},\mathbb{P}_{s,\eta}})$-predictable functions $f : [s,\infty)\times\mathbb{R}\times\Omega \to \mathbb{R}$ satisfying $|||f|||_{s,T} < \infty$ for all $T > s$ where

$$|||f|||_{s,T} := \Big(\int_s^T \int_{\mathbb{R}} \mathbb{E}_{s,\eta}\left[|f(t,x)|^2\right] e^{-|x|}dx\,(t-s)^{1/2}dt\Big)^{1/2}.$$

Note that $\mathcal{P}_s$ is complete w.r.t. the metric $d_{\mathcal{P}_s}(f,g) = \sum_{k=1}^{\infty} 2^{-k}(|||f-g|||_{s,s+k} \wedge 1)$. In Step 2 it will be established that $(X^\epsilon)_{\epsilon\downarrow 0}$ is a Cauchy sequence in $\mathcal{P}_s$ w.r.t. $d_{\mathcal{P}_s}$. In Step 3 the limit X of $(X^\epsilon)_{\epsilon\downarrow 0}$ is shown to satisfy $\bar{X}_t(dx) = X_t(x)dx$ for dt-almost all $t > s$, $\mathbb{P}_{s,\eta}$-almost surely. Finally, in Step 4 we construct a jointly continuous modification X' of X which satisfies $\bar{X}_t(dx) = X'_t(x)dx$ for all $t > s$, $\mathbb{P}_{s,\eta}$-almost surely.

Step 1. For every $T > s$, we have for all $t \in [s,T]$ and $x, x' \in \mathbb{R}$:

$$\begin{aligned}
&|X_t^\epsilon(x) - X_t^\epsilon(x')| \qquad (9.56)\\
&\leq \langle \bar{X}_t, |p_\epsilon(x,.) - p_\epsilon(x',.)|\rangle \leq \langle \bar{X}_t, \mathbf{1}\rangle \sup_{b\in\mathbb{R}} |p_\epsilon(x,b) - p_\epsilon(x',b)|\\
&\leq \langle \bar{X}_t, \mathbf{1}\rangle\, |x-x'|\, \frac{1}{\sqrt{2\pi\epsilon}} \sup_{a,b\in\mathbb{R}} \frac{2|a-b|}{2\epsilon} e^{-\frac{(a-b)^2}{2\epsilon}} \leq K_{\epsilon,T}\, |x-x'|
\end{aligned}$$

where we used the mean value theorem, the elementary estimate $|h|e^{-h^2} < 2$ $(h \in \mathbb{R})$ and $K_{\epsilon,T} := 2(\pi)^{-1/2}\epsilon^{-1}\sup_{t\leq T}\langle \bar{X}_t, \mathbf{1}\rangle$. With help of (9.56) and the continuity of $\bar{X}$ w.r.t. the weak topology one can easily show that $(t,x) \mapsto X_t^\epsilon(x)$ is continuous. Moreover, the map $\nu \mapsto \langle \nu, p_\epsilon(x,.)\rangle$ is clearly continuous and so the variable $X_t^\epsilon(x)$ is $\bar{\mathcal{F}}_{[s,t]}^{\bar{X},\mathbb{P}_{s,\eta}}$-measurable for every $t \geq s$ and $x \in \mathbb{R}$. Then $(X_t^\epsilon(x) : t \geq s, x \in \mathbb{R})$ can easily be shown to be $(\bar{\mathcal{F}}_{[s,t]}^{\bar{X},\mathbb{P}_{s,\eta}})$-predictable. Further, since condition (A) is satisfied, we obtain by the second moment formula in (9.23), Lemma 4.2$(i) \Rightarrow (ii)$ and Lemma 4.4(i) for every $T > s$:

$$\begin{aligned}
&|||X^\epsilon|||_{s,T}^2 = \int_s^T \int_{\mathbb{R}} \mathbb{E}_{s,\eta}[X_t^\epsilon(x)^2] e^{-|x|} dx\, t^{1/2} dt\\
&= \int_s^T \int_{\mathbb{R}} \langle \eta, P_{t-s}p_\epsilon(x,.)\rangle^2 e^{-|x|} dx\, (t-s)^{1/2} dt +\\
&\quad \int_s^T \int_{\mathbb{R}} \int_{\mathbb{R}} \int_s^t \int_{\mathbb{R}} p_{r-s}(z,y) \left(P_{t-r}p_\epsilon(x,.)\right)^2(y) \varrho(drdy)\, \eta(dz) e^{-|x|} dx\, (t-s)^{1/2} dt\\
&= \int_s^T \int_{\mathbb{R}} \frac{1}{2\pi(t-s+\epsilon)} \Big(\int_{\mathbb{R}} e^{-\frac{(x-z)^2}{2(t-s+\epsilon)}} \eta(dz)\Big)^2 e^{-|x|} dx\, (t-s)^{1/2} dt +\\
&\quad \int_s^T \int_{\mathbb{R}} \int_s^t \int_{\mathbb{R}} \frac{(2\pi)^{-1/2}}{\sqrt{r-s}} \Big(\int_{\mathbb{R}} e^{-\frac{(z-y)^2}{2(r-s)}} \eta(dz)\Big) p_{t-r+\epsilon}^2(x,y) \varrho(drdy) e^{-|x|} dx\, (t-s)^{1/2} dt\\
&\leq \frac{1}{2\pi}\langle \eta, \mathbf{1}\rangle^2 \int_s^T \int_{\mathbb{R}} e^{-|x|} dx\, (t-s)^{-1/2} dt +\\
&\quad \frac{\langle \eta, \mathbf{1}\rangle}{(2\pi)^{\frac{3}{2}}} \int_s^T \int_{\mathbb{R}} \int_s^t \frac{1}{\sqrt{r-s}} \frac{1}{(t-r+\epsilon)} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{t-r+\epsilon}} \varrho_1(r,dy) \varrho_2(dr) e^{-|x|} dx (t-s)^{1/2} dt\\
&\leq \tilde{c}_T + \frac{\langle \eta, \mathbf{1}\rangle}{(2\pi)^{\frac{3}{2}}} \int_s^T \int_{\mathbb{R}} \int_s^t \frac{1}{\sqrt{r-s}} \frac{1}{(t-r+\epsilon)} \bar{c}_T (t-r+\epsilon)^{\alpha_1/2} \varrho_2(dr) e^{-|x|} dx\, (t-s)^{1/2} dt\\
&\leq \tilde{c}_T + \tilde{c}'_T \int_s^T \int_{\mathbb{R}} \Big(\frac{1}{(t-s)/2} \int_s^{s+(t-s)/2} \frac{1}{\sqrt{r-s}} \varrho_2(dr)\\
&\qquad + \frac{1}{\sqrt{(t-s)/2}} \int_{s+(t-s)/2}^t \frac{1}{(t-r)^{1-\alpha_1/2}} \varrho_2(dr)\Big) e^{-|x|} dx\, (t-s)^{1/2} dt
\end{aligned}$$

$$\leq \tilde{c}_T + \tilde{c}'' \int_s^T \int_{\mathbb{R}} \Big(\frac{1}{(t-s)^{3/2-\alpha_2}} + \frac{1}{(t-s)^{1/2-(\alpha_1/2+\alpha_2-1)}} \Big) e^{-|x|} dx \, (t-s)^{1/2} dt \leq c_T$$

for some finite constant $c_T > 0$. Hence, $X^\epsilon \in \mathcal{P}_s$ for every $\epsilon > 0$.

Step 2. Again by the second moment formula in (9.23), we have for all $T > s$ and $0 < \epsilon \leq \epsilon' \leq 1$:

$$\begin{aligned}
|||X^\epsilon - X^{\epsilon'}|||_{s,T}^2 &= \int_s^T \int_{\mathbb{R}} \mathbb{E}_{s,\eta}[(X_t^\epsilon(x) - X_t^{\epsilon'}(x))^2] e^{-|x|} dx \, (t-s)^{1/2} dt \\
&= \int_s^T \int_{\mathbb{R}} \langle \eta, P_{t-s}(p_\epsilon(x,.) - p_{\epsilon'}(x,.)) \rangle^2 e^{-|x|} dx \, (t-s)^{1/2} dt \\
&\quad + \int_s^T \int_{\mathbb{R}} \int_{\mathbb{R}} \int_s^t \int_{\mathbb{R}} p_{r-s}(z,y) \\
&\qquad \times \Big(P_{t-r}(p_\epsilon(x,.) - p_{\epsilon'}(x,.)) \Big)^2 (y) \varrho(drdy) \eta(dz) e^{-|x|} dx \, t^{1/2} dt \\
&\leq \int_s^T \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} |p_{t-s+\epsilon}(x,z) - p_{t-s+\epsilon'}(x,z)| \eta(dz) \Big)^2 e^{-|x|} dx \, (t-s)^{1/2} dt \\
&\quad + \int_s^T \int_{\mathbb{R}} \int_{\mathbb{R}} \int_s^t \int_{\mathbb{R}} p_{r-s}(z,y) \\
&\qquad \times \Big(p_{t-r+\epsilon}(x,y) - p_{t-r+\epsilon'}(x,y) \Big)^2 \varrho(drdy) \eta(dz) e^{-|x|} dx \, (t-s)^{1/2} dt \\
&=: I_1 + I_2.
\end{aligned}$$

Using Lemma 4.5(i),

$$\begin{aligned}
I_1 &\leq \int_s^T \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} c \int_{t-s+\epsilon}^{t-s+\epsilon'} \frac{1}{u} p_{2u}(x,z) du \, \eta(dz) \Big)^2 e^{-|x|} dx \, (t-s)^{1/2} dt \\
&\leq \frac{c^2}{4\pi} \int_s^T \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} \int_{t-s+\epsilon}^{t-s+\epsilon'} \frac{1}{u^{3/2}} \, e^{-\frac{(x-z)^2}{2u}} du \, \eta(dz) \Big)^2 e^{-|x|} dx \, (t-s)^{1/2} dt \\
&\leq \frac{c^2}{4\pi} \langle \eta, \mathbf{1} \rangle^2 \int_s^T \int_{\mathbb{R}} \frac{1}{(t-s)^{2(5/8)}} \Big(\int_{t-s+\epsilon}^{t-s+\epsilon'} \frac{1}{u^{7/8}} du \Big)^2 \, e^{-|x|} dx \, (t-s)^{1/2} dt \\
&\leq \tilde{c}_T \, |\epsilon - \epsilon'|^{2(1/8)} = \tilde{c}_T \, |\epsilon - \epsilon'|^{1/4}.
\end{aligned}$$

Pick $\theta \in (0, 2\alpha)$ and split

$$\begin{aligned}
I_2 &= \int_s^T \int_{\mathbb{R}} \int_{\mathbb{R}} \int_s^t \int_{\mathbb{R}} \dots \\
&= \int_s^T \int_{\mathbb{R}} \int_{\mathbb{R}} \int_s^{s+|\epsilon-\epsilon'|^\theta \wedge ((t-s)/2)} \int_{\mathbb{R}} \dots + \int_s^T \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{|\epsilon-\epsilon'|^\theta \wedge ((t-s)/2)}^t \int_{\mathbb{R}} \dots \\
&=: I_{2,1} + I_{2,2}.
\end{aligned}$$

Since $\varrho(dtdx)$ satisfy condition (A) we obtain by Lemmas 4.2(i) $\Rightarrow$ (ii) and 4.4(i):

$$I_{2,1} \leq \int_s^T \int_{\mathbb{R}} \int_s^{s+|\epsilon-\epsilon'|^\theta \wedge ((t-s)/2)} \frac{\langle \eta, \mathbf{1} \rangle}{\sqrt{2\pi(r-s)}} \times$$

$$
\begin{aligned}
& \frac{4}{2\pi\{t-[s+|\epsilon-\epsilon'|^\theta \wedge ((t-s)/2)]\}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{t-r+\epsilon'}} \varrho_1(r,dy)\varrho_2(dr)e^{-|x|}dx\ (t-s)^{1/2}dt \\
\leq\ & \int_s^T \int_{\mathbb{R}} \frac{\langle \eta, \mathbf{1}\rangle}{\sqrt{2\pi}} \frac{4}{2\pi\{(t-s)/2\}}\ \bar{c}_T\ (T-s+\epsilon')^{\alpha_1/2} \\
& \qquad\qquad\qquad\qquad \int_s^{s+|\epsilon-\epsilon'|^\theta} \frac{1}{\sqrt{r-s}} \varrho_2(dr)\ e^{-|x|}dx\ (t-s)^{1/2}dt \\
\leq\ & c'_T \int_s^T \int_{\mathbb{R}} \frac{1}{t-s} \int_s^{s+|\epsilon-\epsilon'|^\theta} \frac{1}{(r-s)^{1-\alpha_1/2}} (r-s)^{\alpha_1/2-1/2} \varrho_2(dr) e^{-|x|}dx (t-s)^{1/2}dt \\
\leq\ & c''_T \int_s^T \int_{\mathbb{R}} \frac{1}{(t-s)^{1/2}} \int_s^{s+|\epsilon-\epsilon'|^\theta} \frac{1}{(r-s)^{1-\alpha_1/2}}\ \varrho_2(dr)\ e^{-|x|}dx\ dt \ \leq\ c'''_T\ |\epsilon-\epsilon'|^{\theta\alpha}
\end{aligned}
$$

where we used $\alpha_2 > 1-\alpha_1/2 \geq \frac{1}{2}$. With help of Lemma 4.6 we also obtain

$$
\begin{aligned}
I_{2,2}\ \leq\ & \int_s^T \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\{s+|\epsilon-\epsilon'|^\theta \wedge ((t-s)/2) - s\}}} \langle \eta, \mathbf{1}\rangle \\
& \times \int_s^t \int_{\mathbb{R}} \Big(p_{t-r+\epsilon}(x,y) - p_{t-r+\epsilon'}(x,y)\Big)^2 \varrho(drdy)\ e^{-|x|}dx\ (t-s)^{1/2}dt \\
\leq\ & \int_s^T \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\{|\epsilon-\epsilon'|^\theta \wedge ((t-s)/2)\}}} \langle \eta, \mathbf{1}\rangle \bar{c}_T |\epsilon-\epsilon'|^\alpha\ e^{-|x|}dx\ (t-s)^{1/2}dt \\
\leq\ & \tilde{c}_T \int_s^T \frac{|\epsilon-\epsilon'|^\alpha}{\sqrt{|\epsilon-\epsilon'|^\theta \wedge ((t-s)/2)}}\ (t-s)^{1/2}dt \\
\leq\ & \tilde{c}_T \int_s^T \Big(|\epsilon-\epsilon'|^{\alpha-\theta/2} + |\epsilon-\epsilon'|^\alpha ((t-s)/2)^{-1/2}\Big)\ (t-s)^{1/2}dt \ \leq\ c_T\ |\epsilon-\epsilon'|^{\alpha-\theta/2}
\end{aligned}
$$

which tends to 0 as $\epsilon, \epsilon' \downarrow 0$ since $\theta \in (0, 2\alpha)$. Hence $(X^\epsilon)_{\epsilon\downarrow 0}$ is a Cauchy sequence in the complete metric space $(\mathcal{P}_s, d_{\mathcal{P}_s})$.

Step 3. Since $\mathbb{R}$ is locally compact, complete and separable, there exists a countable subset $\{f_k\} \equiv \{f_k\}_{k\geq 1}$ of $C_c(S)$ which is separating in $\mathcal{M}_f(\mathbb{R})$ ($\subset \mathcal{M}(\mathbb{R})$); cf. Proposition 2.10. Let X denote the limit of the Cauchy sequence $(X^\epsilon)_{\epsilon\downarrow 0}$. Then we obtain for every $T > s$, $\epsilon > 0$ and $\psi \in C_c(\mathbb{R})$:

$$
\begin{aligned}
& \mathbb{E}_{s,\eta}\Big[\int_s^T |\langle \bar{X}_t, \psi\rangle - \langle X_t, \psi\rangle|^2\ (t-s)^{1/2}dt\Big] \\
& \leq\ 2\ \mathbb{E}_{s,\eta}\Big[\int_s^T |\langle \bar{X}_t, \psi\rangle - \langle X_t^\epsilon, \psi\rangle|^2\ (t-s)^{1/2}dt\Big] \\
& \qquad\qquad\qquad\qquad +2\ \mathbb{E}_{s,\eta}\Big[\int_s^T |\langle X_t^\epsilon, \psi\rangle - \langle X_t, \psi\rangle|^2\ (t-s)^{1/2}dt\Big] \\
& \leq\ 2\ \mathbb{E}_{s,\eta}\Big[\int_s^T |\langle \bar{X}_t, \psi - P_\epsilon\psi\rangle|^2\ (t-s)^{1/2}dt\Big] \qquad\qquad (9.57) \\
& \quad +2 \int_{\mathbb{R}} \psi^2(x)dx\ \mathbb{E}_{s,\eta}\Big[\int_s^T \int_{\mathrm{supp}(\psi)} (X_t^\epsilon(x) - X_t(x))^2\ dx\ (t-s)^{1/2}dt\Big].
\end{aligned}
$$

The first summand on the r.h.s. of (9.57) tends to 0 as $\epsilon \downarrow 0$ by dominated convergence and the strong continuity of the heat semigroup (P_t). By Step 2 and Fubini's theorem for non-negative measurable integrands, the second summand tends to 0 as $\epsilon \downarrow 0$, too. So the l.h.s. of (9.57) vanishes and we infer $\langle \bar{X}_t, f_k\rangle = \langle X_t, f_k\rangle$ for all $k \geq 1$ and dt-almost all $t > s$, $\mathbb{P}_{s,\eta}$-almost surely. Consequently, since $\{f_k\}$ is separating, $\bar{X}_t(dx) = X_t(x)dx$ holds for dt-almost all $t > s$, $\mathbb{P}_{s,\eta}$-almost surely.

Step 4. Let again X denote the limit of the Cauchy sequence (X^ϵ) from Step 2. We first prove the following lemma.

Lemma 9.29 *For $n \geq 1$, $s \leq t_0 < t_1 \leq t, t'$ and $x, x' \in \mathbb{R}$ we have:*

$$\mathbb{E}_{s,\eta}\Big[|\langle X_{t_0}, p_{t-t_0}(x,.)\rangle - \langle X_{t_0}, p_{t'-t_0}(x',.)\rangle|^{2n}\Big] \;\leq\; c_{n,t_0,t_1-t_0}\,(|t-t'|^{2n} + |x-x'|^{2n}).$$

If M denotes the continuous orthogonal martingale measure from Proposition 9.22, then we also obtain for $m \geq 1$, $s < t_0 \leq t, t' \leq T$ and $x, x' \in \mathbb{R}$:

$$\begin{aligned}\mathbb{E}_{s,\eta}\Big[\Big|\int_{t_0}^{t}\int_{\mathbb{R}} p_{t-r}(x,y)M(drdy) - \int_{t_0}^{t'}\int_{\mathbb{R}} p_{t'-r}(x',y)M(drdy)\Big|^{2m}\Big]&\\ \leq\; c_{t_0,m,T}\,(|t-t'|^{\alpha} + |x-x'|^{2\alpha})^m.&\end{aligned}$$

Proof Assume w.l.o.g. $t \leq t'$ and recall our convention $p_t \equiv 0$ for $t < 0$. The first inequality in Lemma 9.29 follows from

$$\begin{aligned}
&\mathbb{E}_{s,\eta}\Big[|\langle X_{t_0}, p_{t-t_0}(x,.)\rangle - \langle X_{t_0}, p_{t'-t_0}(x',.)\rangle|^{2n}\Big]\\
&\leq\; 2^{2n}\,\mathbb{E}_{s,\eta}\Big[\langle X_{t_0}, |p_{t-t_0}(x,.) - p_{t'-t_0}(x,.)|\rangle^{2n}\Big]\\
&\qquad\qquad +2^{2n}\,\mathbb{E}_{s,\eta}\Big[\langle X_{t_0}, |p_{t'-t_0}(x,.) - p_{t'-t_0}(x',.)|\rangle^{2n}\Big]\\
&\leq\; 2^{2n}\,\mathbb{E}_{s,\eta}\Big[\Big(\int_{\mathbb{R}} X_{t_0}(y)\; c\int_{t-t_0}^{t'-t_0}\frac{1}{u}p_{2u}(x,y)du\,dy\Big)^{2n}\Big]\\
&\quad +2^{2n}\,\frac{1}{(2\pi(t_1-t_0))^n}\mathbb{E}_{s,\eta}\Big[\Big(\int_{\mathbb{R}} X_{t_0}(y)\;|x-x'|\sup_{a\in\mathbb{R}}\frac{2|a-y|}{2(t'-t_0)}e^{-\frac{(a-y)^2}{2(t'-t_0)}}dy\Big)^{2n}\Big]\\
&\leq\; 2^{2n}c^{2n}\frac{1}{(t_1-t_0)^{3n}}\,\mathbb{E}_{s,\eta}\Big[\langle X_{t_0}, \frac{1}{\sqrt{2\pi}}\,2\,|t-t'|\rangle^{2n}\Big]\\
&\qquad\qquad +\,2^{2n}\,\frac{1}{(2\pi(t_1-t_0))^n}\frac{2^n}{(t_1-t_0)^n}|x-x'|^{2n}\,\mathbb{E}_{s,\eta}\Big[\langle X_{t_0}, \mathbf{1}\rangle^{2n}\Big]\\
&\leq\; c_{n,t_0,t_1-t_0}\,(|t-t'|^{2n} + |x-x'|^{2n})
\end{aligned}$$

where we used Lemma 4.5(i), the mean value theorem for differentials, the elementary estimate $|h|\,e^{-h^2} < 2$ ($h \in \mathbb{R}$) and Theorem 9.5. The second inequality in Lemma 9.29 can be obtained as follows. Using the Burkholder-Davis-Gundy inequality (Theorem 3.28) applied to the martingale $u \mapsto \int_{t_0}^{u}\int_{\mathbb{R}}(p_{t-r}(x,y) - p_{t'-r}(x',y))M(drdy)$, $t_0 \leq u \leq t'$, we obtain

$$\mathbb{E}_{s,\eta}\Big[|\int_{t_0}^{t}\int_{\mathbb{R}} p_{t-r}(x,y)M(drdy) - \int_{t_0}^{t'}\int_{\mathbb{R}} p_{t'-r}(x',y)M(drdy)|^{2m}\Big]$$

$$\begin{aligned}
&= \mathbb{E}_{s,\eta}\Big[\big|\int_{t_0}^{t'}\int_{\mathbb{R}} (p_{t-r}(x,y) - p_{t'-r}(x',y))M(drdy)\big|^{2m}\Big] \\
&\leq c_m \Big[\Big(\int_{t_0}^{t'}\int_{\mathbb{R}} (p_{t-r}(x,y) - p_{t'-r}(x',y))^2\langle M\rangle(drdy)\Big)^m\Big] \\
&= c_m\, \mathbb{E}_{s,\eta}\Big[\Big(\int_{t_0}^{t'}\int_{\mathbb{R}} (p_{t-r}(x,y) - p_{t'-r}(x',y))^2 C_{[\bar{X},\varrho]}(drdy)\Big)^m\Big].
\end{aligned}$$

By Theorem 9.14, i.e. by (ii) of Definition 9.13, we may continue

$$\begin{aligned}
&= c_m\, \mathbb{E}_{s,\eta}\Big[\Big(\int_{t_0}^{t'}\int_{\mathbb{R}} \lim_{K\uparrow\infty}\Big((p_{t-r}(x,y) - p_{t'-r}(x',y))^2 \wedge K\Big) C_{[\bar{X},\varrho]}(drdy)\Big)^m\Big] \\
&\leq c_m \liminf_{K\uparrow\infty}\ \mathbb{E}_{s,\eta}\Big[\Big(\int_{t_0}^{t'}\int_{\mathbb{R}} \Big((p_{t-r}(x,y) - p_{t'-r}(x',y))^2 \wedge K\Big) C_{[\bar{X},\varrho]}(drdy)\Big)^m\Big] \\
&= c_m \liminf_{K\uparrow\infty}\ \mathbb{E}_{s,\eta}\Big[\Big(\lim_{\epsilon\downarrow 0}\int_{t_0}^{t'}\int_{\mathbb{R}}\int_{\mathbb{R}} \Big((p_{t-r}(x,y) - p_{t'-r}(x',y))^2 \wedge K\Big) \\
&\qquad\qquad \times p_\epsilon(y,z)\bar{X}_r(dz)\varrho(drdy)\Big)^m\Big] \\
&\leq c_m\, \mathbb{E}_{s,\eta}\Big[\Big(\lim_{\epsilon\downarrow 0}\int_{t_0}^{t'}\int_{\mathbb{R}}\int_{\mathbb{R}} (p_{t-r}(x,y) - p_{t'-r}(x',y))^2 p_\epsilon(y,z)\bar{X}_r(dz)\varrho(drdy)\Big)^m\Big].
\end{aligned}$$

The truncation "$\wedge K$" guaranteed that Theorem 9.14 could be applied. By means of Hölder's inequality ($\frac{m-1}{m} + \frac{1}{m} = 1$), Fatou's lemma, Lemma 9.28 and Lemma 4.6 we may continue

$$\begin{aligned}
&\leq c_m \Big(\int_{t_0}^{t'}\int_{\mathbb{R}} (p_{t-r}(x,y) - p_{t'-r}(x',y))^2\varrho(drdy)\Big)^{m-1} \\
&\qquad \times \liminf_{\epsilon\downarrow 0}\int_{t_0}^{t'}\int_{\mathbb{R}} (p_{t-r}(x,y) - p_{t'-r}(x',y))^2 \mathbb{E}_{s,\eta}[\langle \bar{X}_r, p_\epsilon(y,.)\rangle^m]\varrho(drdy) \\
&\leq c_{t_0,m} \Big(\int_{t_0}^{t'}\int_{\mathbb{R}} (p_{t-r}(x,y) - p_{t'-r}(x',y))^2\varrho(drdy)\Big)^m \\
&\leq c_{t_0,T,m}\, (|t-t'|^\alpha + |x-x'|^{2\alpha})^m.
\end{aligned}$$

This completes the proof of Lemma 9.29. □

Fix $t_0 > s$. Corollary 9.22 and Step 3 yield

$$\begin{aligned}
\langle X_{t_0}, P_{t-t_0}\psi\rangle &= \langle \eta, P_{t_0-s}P_{t-t_0}\psi\rangle + \int_s^{t_0}\int_{\mathbb{R}} P_{t_0-r}P_{t-t_0}\psi(y)M(drdy) \\
\langle X_t, \psi\rangle &= \langle \eta, P_{t-s}\psi\rangle + \int_s^t\int_{\mathbb{R}} P_{t-r}\psi(y)M(drdy)
\end{aligned}$$

$\mathbb{P}_{s,\eta}$-almost surely, for dt-almost all $t \geq t_0$ and every $\psi \in C_b^2(\mathbb{R})$. Consequently,

$$\langle X_t, \psi\rangle = \langle X_{t_0}, P_{t-t_0}\psi\rangle + \int_{t_0}^t\int_{\mathbb{R}} P_{t-r}\psi(y)M(drdy) \tag{9.58}$$

holds $\mathbb{P}_{s,\eta}$-almost surely, for dt-almost all $t \geq t_0$ and every $\psi \in C_b^2(\mathbb{R})$. In particular,

$$\langle X_t, p_\epsilon(x,.)\rangle = \langle X_{t_0}, p_{t-t_0+\epsilon}(x,.)\rangle + \int_{t_0}^t \int_{\mathbb{R}} p_{t-r+\epsilon}(x,y) M(drdy) \tag{9.59}$$

holds $\mathbb{P}_{s,\eta}$-almost surely, for dt-almost all $t \geq t_0$ and every $x \in \mathbb{R}$ and $\epsilon > 0$. Moreover,

$$\begin{aligned} X_t(x) &= \lim_{\epsilon' \downarrow 0} \langle X_t, p_{\epsilon'}(x,.)\rangle \\ \langle X_{t_0}, p_{t-t_0}(x,.)\rangle &= \lim_{\epsilon' \downarrow 0} \langle X_{t_0}, p_{t-t_0+\epsilon'}(x,.)\rangle \\ \int_{t_0}^t \int_{\mathbb{R}} p_{t-r}(x,y) M(drdy) &= \lim_{\epsilon' \downarrow 0} \int_{t_0}^t \int_{\mathbb{R}} p_{t-r+\epsilon'}(x,y) M(drdy) \end{aligned} \tag{9.60}$$

hold $\mathbb{P}_{s,\eta}$-almost surely (for some subsequence $(\epsilon') \subset (\epsilon)$), for $dtdx$-almost all $(t,x) \in [t_0,\infty) \times \mathbb{R}$. The first line in (9.60) is justified by $d_{\mathcal{P}_s}(X, X^\epsilon) \to 0$, the second line can be obtained easily and the third line follows from the second inequality in Lemma 9.29. We deduce from (9.59) and (9.60) that

$$X_t(x) = \langle X_{t_0}, p_{t-t_0}(x,.)\rangle + \int_{t_0}^t \int_{\mathbb{R}} p_{t-r}(x,y) M(drdy) \tag{9.61}$$

holds $\mathbb{P}_{s,\eta}$-almost surely, for $dtdx$-almost all $(t,x) \in [t_0,\infty) \times \mathbb{R}$. Let us denote the r.h.s. of (9.61) by $X_t''(x)$. Then, for every $\gamma \in (0, \alpha/2)$, X'' has a locally jointly Hölder-γ-continuous modification X' on $(t_1,\infty) \times \mathbb{R}$ ($\forall t_1 > t_0 > s$), and hence on $(s,\infty) \times \mathbb{R}$. This follows from Kolmogorov's theorem (cf. [Wal86], Corollary 1.2) and Lemma 9.29 with n, m sufficiently large such that $n \geq 2$ and $(m\alpha - 2)/(2m) > \gamma$. Also, X' is a modification of X in the sense that $X_t'(x) = X_t(x)$ $\mathbb{P}_{s,\eta}$-almost surely, for $dtdx$-almost all $(t,x) \in (s,\infty) \times \mathbb{R}$. Since X is predictable and X' is continuous, the mapping $(t,x,\omega) \mapsto |X_t(x,\omega) - X_t'(x,\omega)|$ is $\mathcal{B}|_{[s,\infty)} \times \mathcal{B} \times \mathcal{F}$-measurable. Hence we can apply Fubini's theorem for non-negative measurable integrands. Thus,

$$\mathbb{E}_{s,\eta}\Big[\int_s^\infty \int_{\mathbb{R}} |X_t(x) - X_t'(x)| dxdt\Big] = \int_s^\infty \int_{\mathbb{R}} \mathbb{E}_{s,\eta}[|X_t(x) - X_t'(x)|] dxdt = 0.$$

In particular, $X_t'(x) = X_t(x)$ for $dtdx$-almost all $(t,x) \in (s,\infty) \times \mathbb{R}$, $\mathbb{P}_{s,\eta}$-almost surely. So, if we define $\bar{X}_t'(dx) := X_t'(x)dx$ for all $t \geq s$, we have by Step 3 $\bar{X}_t(dx) = \bar{X}_t'(dx)$ for dt-almost all $t > s$, $\mathbb{P}_{s,\eta}$-almost surely. By dominated convergence and the joint continuity of X' we obtain continuity of $(\langle \bar{X}_t', \psi\rangle : t > s)$ for all $\psi \in C_c(\mathbb{R})$. Combining this with the weak continuity of $\bar{X}$ and the fact that $C_c(\mathbb{R})$ is separating in $\mathcal{M}_f(\mathbb{R})$ yields $\bar{X}_t(dx) = \bar{X}_t'(dx)$ for all $t > s$, $\mathbb{P}_{s,\eta}$-almost surely. That is, $(X_t'(x) : t > s, x \in \mathbb{R})$ is $\mathbb{P}_{s,\eta}$-almost surely a (jointly continuous) density field of $(\bar{X}_t(dx) : t > s)$. This completes the proof of Theorem 9.27.

9.9 SPDE for the density field

In this section we characterize the jointly continuous density field from the previous section as the unique solution to a certain SPDE. Let the catalyst $\varrho(dtdx)$ satisfy condition (A) and $\bar{X} = (\bar{X}_t : t \geq s)$ be the corresponding weakly continuous catalytic SBM with initial state $\bar{X}_s = \eta \in \mathcal{M}_f(\mathbb{R})$. From Theorem 9.27 we know that $(\bar{X}_s : t > s)$ possesses a jointly continuous density field $X = (X_t(.) : t > s)$. Let

$$C^+_{int}(\mathbb{R}^d) := \{\psi \in C^+(\mathbb{R}^d) : \langle \psi, \mathbf{1} \rangle < \infty\}$$

be equipped with the induced weak topology. We will prove that X is the weakly unique weak $C^+_{int}(\mathbb{R})$-valued solution to the following SPDE:

$$\begin{aligned} \frac{\partial}{\partial t} X_t(x) &= \frac{1}{2}\Delta X_t(x) + \sqrt{X_t(x)}\frac{\partial^2}{\partial t \partial x} W^{\varrho}(t,x) \qquad (t > s,\ x \in \mathbb{R}) \\ X_s &= \eta(dx) \end{aligned} \tag{9.62}$$

where $\dot{W}^{\varrho}(t,x) = \frac{\partial^2}{\partial t \partial x} W^{\varrho}(t,x)$ is a time-space white noise with intensity measure $\varrho(dtdx)$. In Definition 5.20 we specified solutions of the stochastic heat equation. Here we use a slightly different definition: If one can find any continuous orthogonal martingale measure $W^{\varrho} = [W^{\varrho}, \Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}]$ with quadratic variation measure $\varrho(dtdx)$ and a $C^+_{int}(\mathbb{R})$-valued continuous $(\mathcal{F}_t)$-predictable process $Y = (Y_t(.) : t > s)$ on $[\Omega, \mathcal{F}, \mathbb{P}]$ such that

$$\langle Y_t(.), \psi \rangle = \langle \eta, \psi \rangle + \int_s^t \langle Y_r(.), \frac{1}{2}\Delta\psi \rangle dr + \int_s^t \int_{\mathbb{R}} \sqrt{Y_r(y)}\psi(y) W^{\varrho}(drdy)$$

holds for all $t > s$ and $\psi \in C^{\infty}_c(\mathbb{R}^d)$, $\mathbb{P}$-almost surely, then Y is called *weak $C^+_{int}(\mathbb{R})$-valued solution* to SPDE (9.62) with initial condition $\eta \in \mathcal{M}_f(\mathbb{R})$. The solution is said to be *weakly unique* if any two solutions (which might be defined on different probability spaces) coincide in law.

Theorem 9.30 [SPDE FOR THE DENSITY FIELD] *Let $\varrho(dtdx)$ satisfy condition* (A) *and $\bar{X} = [\bar{X}, \mathbb{P}_{s,\eta} : s \geq 0, \eta \in \mathcal{M}_f(\mathbb{R})]$ denote the corresponding canonical continuous catalytic SBM. Pick $s \geq 0$ and $\eta \in \mathcal{M}_f(\mathbb{R})$. By Theorem 9.27, $(\bar{X}_t(dx) : t > s)$ possesses a jointly continuous density field $X = (X_t(.) : t > s)$ under $\mathbb{P}_{s,\eta}$. This density field is the weakly unique weak solution to SPDE (9.62) with initial condition η.*

Proof Recall from Theorem 9.21 that $\bar{X} = (\bar{X}_t : t \geq s)$ under $\mathbb{P}_{s,\eta}$ is the unique solution to the martingale problem $\mathrm{MP}_{s,\eta}$ posed in Definition 9.20. Pick a continuous orthogonal martingale measure $\tilde{W}^{\varrho}$ with quadratic variation measure $\langle \tilde{W}^{\varrho} \rangle(dtdx) = \varrho(dtdx)$ and let M be the martingale measure from Corollary 9.22. Suppose $\tilde{W}^{\varrho}$ is independent of M; if necessary, consider an enlargement of $\bar{X}$'s domain $[\Omega, \mathcal{F}, (\bar{\mathcal{F}}^{\bar{X},\mathbb{P}_{s,\eta}}_{[s,t]}), \mathbb{P}_{s,\eta}]$. Note that $\langle M \rangle(dtdx)(= C_{[\bar{X},\varrho]}(dtdx)) = X_t(x)\varrho(dtdx)$ holds, where $C_{[\bar{X},\varrho]}(dtdx)$ is the collision measure of $\bar{X}$ and ϱ (recall that $\bar{X}$'s density field X is jointly continuous). Also, $(X_t(.) : t > s)$ is a $C^+_{int}(\mathbb{R})$-valued continuous $(\bar{\mathcal{F}}^{\bar{X},\mathbb{P}_{s,\eta}}_{[s,t]})$-predictable process.[38] Then proceed as in the proof of Proposition 6.4 in order to show that $(X_t(.) : t > s)$ solves SPDE (9.62) weakly.

[38] Let $(X_t(x) : t > s, x \in \mathbb{R})$ be the density field from Theorem 9.27. Clearly, $x \mapsto X_t(\omega, x)$ is in $C^+_{int}(\mathbb{R})$ $\forall \omega \in \Omega$, i.e. $X_t : \Omega \to C^+_{int}(\mathbb{R})$. Also, $X_t^{-1}(H) = \bar{X}_t^{-1}(\bar{H})$ for $H \in \mathcal{B}(C^+_{int}(\mathbb{R}))$, $\bar{H} \in \mathcal{B}(\mathcal{M}_f(\mathbb{R}))$ with $H = \bar{H} \cap C^+_{int}(\mathbb{R})$ (if necessary, use the convention $X :\equiv 0 \equiv: \bar{X}$ on an exceptional null set Ω_0). Hence, X_t is $[\bar{\mathcal{F}}^{\bar{X},\mathbb{P}_{s,\eta}}_{[s,t]}, \mathcal{B}(C^+_{int}(\mathbb{R}))]$-measurable since $\bar{X}_t$ is $[\bar{\mathcal{F}}^{\bar{X},\mathbb{P}_{s,\eta}}_{[s,t]}, \mathcal{B}(\mathcal{M}_f(\mathbb{R}))]$-measurable. In particular, $(X_t(.) : t > s)$ is an $(\bar{\mathcal{F}}^{\bar{X},\mathbb{P}_{s,\eta}}_{[s,t]})$-adapted $C^+_{int}(\mathbb{R})$-valued continuous process. Using the joint continuity, one easily deduces that $(X_t(x) : t > s, x \in \mathbb{R})$ is also $(\bar{\mathcal{F}}^{\bar{X},\mathbb{P}_{s,\eta}}_{[s,t]})$-predictable.

The weak uniqueness follows from the uniqueness of solutions to the martingale problem posed in Definition 9.20 (recall Theorem 9.21). Indeed, if $Y = [(Y_t(.) : t > s), W^\varrho, \Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}]$ is a weak solution to SPDE (9.62) with initial condition η, then

$$M_{s,t}(\psi) := \langle Y_t(.), \psi\rangle - \langle \eta, \psi\rangle - \int_s^t \langle Y_r(.), \frac{1}{2}\Delta\psi\rangle dr = \int_s^t \int_{\mathbb{R}} \psi(y)\sqrt{Y_r(y)}W^\varrho(drdy), \quad t > s$$

provides a continuous square-integrable $(\mathcal{F}_{[s,t]})$-martingale with quadratic variation process

$$\begin{aligned}\langle M_{s,.}(\psi)\rangle_t &= \int_s^t \int_{\mathbb{R}} \psi^2(y) Y_r(y)\langle W^\varrho\rangle(drdy) \\ &= \int_s^t \int_{\mathbb{R}} \psi^2(y) Y_r(y)\varrho(drdy) = \int_s^t \int_{\mathbb{R}} \psi^2(y) C_{[\bar{Y},\varrho]}(drdy), \quad t > s\end{aligned}$$

where $\bar{Y}_t(dx) := Y_t(x)dx$. So the law of $\bar{Y}$ ($\simeq$ law of Y)[39] solves the martingale problem $\mathrm{MP}_{s,\eta}$ posed in Definition 9.20 and is hence unique. □

9.10 Laplace transforms of the density field

In Theorem 9.27 we have seen that the 1-dimensional weakly continuous catalytic SBM $\bar{X}$ possesses a jointly continuous density field X, provided the catalyst $\varrho(dtdx)$ satisfies condition (A). This section is devoted to the Laplace transform of $X_t(x)$ ($t > s$, $x \in \mathbb{R}$). More generally, we will prove that for every $0 \le s < t$ and $\eta \in \mathcal{M}_f(\mathbb{R})$:

$$\mathbb{E}_{s,\eta}\Big[e^{-\langle \nu, X_t\rangle}\Big] = e^{-\langle \eta, U_{s,t}\nu\rangle}, \qquad \nu \in \mathcal{M}_f(\mathbb{R}) \tag{9.63}$$

where $(U_{s,t}\nu(x) : s \in [0,t), x \in \mathbb{R})$ is the unique non-negative solution to the formal BPDE

$$\begin{aligned}\frac{\partial}{\partial s}u(s,t,x) &= \frac{1}{2}\Delta u(s,t,x) - u^2(s,t,x)\frac{\varrho(dsdx)}{dsdx}(s,x) \qquad (s \in [0,t),\ x \in \mathbb{R}) \\ u(t,t) &= \nu(dx).\end{aligned} \tag{9.64}$$

In view of (7.16), we regard BPDE (9.64) as

$$u(s,t,x) = P_{t-s}\nu(x) - \frac{1}{2}\int_s^t \int_{\mathbb{R}^d} p_{r-s}(x,y)u^2(r,t,y)\varrho(drdy), \quad s \in [0,t), x \in \mathbb{R}^d \tag{9.65}$$

where $P_t\nu(x) := \int_{\mathbb{R}} p_t(x,y)\nu(dy)$ for all $t > 0$, $x \in \mathbb{R}$ and $\nu \in \mathcal{M}_f(\mathbb{R})$. Note that the map $\lambda \mapsto L_{X_t(x)}(\lambda) := \mathbb{E}_{s,\eta}[e^{-\langle \lambda\delta_x, X_t(.)\rangle}] = \mathbb{E}_{s,\eta}[e^{-\lambda X_t(x)}]$ from $\mathbb{R}_+$ to $[0,1]$ is just the Laplace transform of the random variable $X_t(x)$ under $\mathbb{P}_{s,\eta}$.

Theorem 9.31 [Laplace transform] *Let $\varrho(dtdx)$ satisfy condition* (A) *and $\bar{X} = [\bar{X}, \mathbb{P}_{s,\eta} : s \ge 0, \eta \in \mathcal{M}_f(\mathbb{R})]$ be the corresponding canonical continuous catalytic SBM. Pick $s \ge 0$ and $\eta \in \mathcal{M}_f(\mathbb{R})$. The jointly continuous density field $(X_t : t > s)$ of $(\bar{X}_t : t > s)$, which exists under $\mathbb{P}_{s,\eta}$ by Theorem 9.27, satisfies (9.63) for all $t > s$ and $\eta \in \mathcal{M}_f(\mathbb{R})$.*

[39] Note that $\bar{Y}$ and Y induce laws on $C([s,\infty), \mathcal{M}_f(\mathbb{R}))$, resp. $C((s,\infty), C^+_{int}(\mathbb{R}))$ (recall Remark 3.7). These laws can clearly be identified since $\bar{Y}_.(dx) \equiv Y_.(x)dx$ and both $\mathcal{M}_f(\mathbb{R})$ and $C^+_{int}(\mathbb{R})$ are furnished with the weak topology (resp. induced weak topology).

The remainder of this section is devoted to the proof of Theorem 9.31. The key is:

Lemma 9.32 *Pick $\nu \in \mathcal{M}_f(\mathbb{R})$ and let $\mu(dtdx)$ satisfy condition* (B)*. Then the integral equation (9.65) with ϱ replaced by μ has a unique non-negative (measurable) solution $(U_{s,t}\nu(x) : 0 \le s < t, x \in \mathbb{R})$ if the following two assertions hold for all $0 \le s < t$:*

(i) the map $x \mapsto P_{t-s}\nu(x)$ is bounded,

(ii) $\int_s^t \int_{\mathbb{R}} p_{r-s}(x,y)(P_{t-r}\nu)^2(y)\mu(drdy) < \infty$ for dx-a.a. $x \in \mathbb{R}$.

In view of moment formula (8.11), the statement of Lemma 9.32 is known from Proposition 1.2 of [Kle00a]. We next show that (i) and (ii) (with μ replaced by ϱ, and even for all $x \in \mathbb{R}$) hold whenever $\varrho(dtdx)$ satisfies condition (A). Assertion (i) is a consequence of

$$P_{t-s}\nu(x) = \int_{\mathbb{R}} p_{t-s}(x,y)\nu(dy) \le \frac{1}{\sqrt{2\pi(t-s)}}\langle \nu, \mathbf{1}\rangle < \infty \qquad \forall s < t, x \in \mathbb{R}.$$

Assertion (ii) follows from

$$\begin{aligned}
&\int_s^t \int_{\mathbb{R}} p_{r-s}(x,y)(P_{t-r}\nu)^2(y)\varrho(drdy) \\
&\quad\le \int_s^t \frac{1}{\sqrt{2\pi(r-s)}}\frac{1}{\sqrt{2\pi(t-r)}}\langle \nu, \mathbf{1}\rangle \int_{\mathbb{R}}\int_{\mathbb{R}} p_{t-r}(z,y)\nu(dz)\varrho_1(r,dy)\varrho_2(dr) \\
&\quad\le \frac{1}{(2\pi)^{3/2}}\langle \nu, \mathbf{1}\rangle \int_s^t \frac{1}{\sqrt{r-s}}\frac{1}{t-r}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-\frac{(z-y)^2}{t-r}}\varrho_1(r,dy)\nu(dz)\varrho_2(dr) \\
&\quad\le \frac{1}{(2\pi)^{3/2}}\langle \nu, \mathbf{1}\rangle^2 \int_s^t \frac{1}{\sqrt{r-s}}\frac{1}{t-r}\bar{c}_t(2(t-r))^{\alpha_1/2}\varrho_2(dr) \\
&\quad\le c_{T,\nu}\frac{1}{((t-s)/2)^{1-\alpha_1/2}}\int_s^{s+(t-s)/2}\frac{1}{\sqrt{r-s}}\varrho_2(dr) \\
&\qquad\qquad +c_{T,\nu}\frac{1}{\sqrt{(t-s)/2}}\int_{s+(t-s)/2}^t \frac{1}{(t-r)^{1-\alpha_1/2}}\varrho_2(dr) \ < \infty \qquad \forall s < t, x \in \mathbb{R}
\end{aligned}$$

for which we used Lemmas 4.2$(i) \Rightarrow (ii)$ and 4.4(i). By Lemma 9.32 we thus have for every $\nu \in \mathcal{M}_f(\mathbb{R})$ a unique non-negative solution $U_{\cdot,t}\nu(.)$ to equation (9.65). If $\nu \in \mathcal{M}_f(\mathbb{R})$, then $\nu^\epsilon(dx) := \int_{\mathbb{R}} p_\epsilon(x,y)\nu(dy)dx$ is an element of $\mathcal{M}_f(\mathbb{R})$, too. The next lemma shows that $U_{s,t}(\nu)(.)$ can be approximated by $U_{s,t}\nu^\epsilon(.)$ as $\epsilon \downarrow 0$.

Lemma 9.33 *Pick $\nu \in \mathcal{M}_f(\mathbb{R})$, let $\varrho(dtdx)$ satisfy condition* (A) *and define $\nu^\epsilon(dx) := \int_{\mathbb{R}} p_\epsilon(x,y)\nu(dy)dx$, $\epsilon > 0$. Then there are unique non-negative (measurable) solutions $U_{\cdot,t}\nu(.)$ and $U_{\cdot,t}\nu^\epsilon(.)$ to equation (9.65) with final state ν, respectively ν^ϵ ($\forall \epsilon > 0$). Moreover, for all $0 \le s < t$:*

$$\lim_{\epsilon \downarrow 0} \|U_{s,t}\nu(.) - U_{s,t}\nu^\epsilon(.)\|_\infty = 0.$$

Proof The statement on the existence and the uniqueness of solutions was just established as a consequence of Lemma 9.32. The remainder of the proof is devoted to the approximation of $U_{s,t}\nu(.)$ by $U_{s,t}\nu^\epsilon(.)$. Fix some $t_0 \in (s,t)$.

Step 1. As the first step we prove the following inequality (for any $\theta \in (0,2)$):

$$\begin{aligned} &\|U_{s,t}(.|\nu) - U_{s,t}(.|\nu^\epsilon)\|_\infty^2 \\ &\quad\le \bar{c}_{t,\nu,t-t_0}\Big(\epsilon^\theta + \int_{t_0}^t (t-r)^{\alpha/2}\|U_{r,t}(.|\nu) - U_{r,t}(.|\nu^\epsilon)\|_\infty^2 dr\Big) \quad \forall \epsilon \in (0,1]. \end{aligned} \tag{9.66}$$

We intend an application of the Gronwall-type Lemma 4.11. Using the elementary inequality $U_{\tau,t}(.|\nu) \le S_{t-\tau}\nu(.)$ we get for all $\tau \in [s,t_0]$

$$\begin{aligned} &\|U_{\tau,t}\nu(.) - U_{\tau,t}\nu^\epsilon(.)\|_\infty^2 \\ &\le\ 2\sup_{x\in\mathbb{R}}|P_{t-\tau}\nu(x) - P_{t-\tau}\nu^\epsilon(x)|^2 \\ &\quad +2\sup_{x\in\mathbb{R}}\Big(\frac{1}{2}\int_\tau^t\int_{\mathbb{R}} p_{r-\tau}(x,y)|U_{r,t}\nu(y)^2 - U_{r,t}\nu^\epsilon(y)^2|\varrho(drdy)\Big)^2 \\ &\le\ 2\sup_{x\in\mathbb{R}}\Big(\int_{\mathbb{R}}|p_{t-\tau}(x,y) - p_{t-\tau+\epsilon}(x,y)|\nu(dy)\Big)^2 + \\ &\quad +2\sup_{x\in\mathbb{R}}\Big(\frac{1}{2}\int_\tau^t\int_{\mathbb{R}} p_{r-\tau}(x,y)(P_{t-r}\nu(y) + P_{t-r}\nu^\epsilon(y)) \\ &\qquad\qquad\qquad \times|U_{r,t}\nu(y) - U_{r,t}\nu^\epsilon(y)|\varrho(drdy)\Big)^2 \\ &=:\ J_1^\epsilon + J_2^\epsilon. \end{aligned}$$

For every $\theta \in (0,2)$ we can estimate J_1^ϵ by

$$\begin{aligned} &2\sup_{x\in\mathbb{R}}\Big(\int_{\mathbb{R}}\int_{t-\tau}^{t-\tau+\epsilon}\frac{1}{u}p_{2u}(x,y)du\nu(dy)\Big)^2 \\ &\le\ 2\frac{1}{4\pi}\langle\nu,\mathbf{1}\rangle^2\frac{1}{(t-\tau)^{1+\theta}}\Big(\int_{t-\tau}^{t-\tau+\epsilon}\frac{1}{u^{1-\theta/2}}du\Big)^2 \le\ \tilde{c}\frac{1}{(t-t_0)^{1+\theta}}\ \epsilon^\theta \end{aligned}$$

where we used Lemma 4.5(i). J_2^ϵ can be estimated by

$$\begin{aligned} &\frac{1}{8\pi^2}\sup_{x\in\mathbb{R}}\Big(\int_\tau^{t_0+\frac{t-t_0}{2}}\frac{1}{\sqrt{r-\tau}}\frac{2\langle\nu,\mathbf{1}\rangle}{\sqrt{t-r}}\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{2(r-\tau)}}\varrho_1(r,dy)\|U_{r,t}\nu(.) - U_{r,t}\nu^\epsilon(.)\|_\infty\varrho_2(dr)\Big)^2 \\ &\quad+\frac{1}{4\pi}\sup_{x\in\mathbb{R}}\Big(\int_{t_0+\frac{t-t_0}{2}}^t\frac{1}{\sqrt{r-\tau}}\Big\{\int_{\mathbb{R}}\int_{\mathbb{R}} p_{t-r}(y,a)\varrho_1(r,dy)\nu(da) \\ &\qquad\qquad+\int_{\mathbb{R}}\int_{\mathbb{R}} p_{t-r+\epsilon}(y,a)\varrho_1(r,dy)\nu(da)\Big\}\|U_{r,t}\nu(.) - U_{r,t}\nu^\epsilon(.)\|_\infty\varrho_2(dr)\Big)^2 \\ &\le\ \frac{\langle\nu,\mathbf{1}\rangle^2}{2\pi^2}\frac{2}{t-t_0}\Big(\int_\tau^{t_0+\frac{t-t_0}{2}}\bar{c}_t\frac{(2(r-\tau))^{\alpha_1/2}}{(r-\tau)^{1/2}}\|U_{r,t}\nu(.) - U_{r,t}\nu^\epsilon(.)\|_\infty\varrho_2(dr)\Big)^2 \\ &\quad+\frac{\langle\nu,\mathbf{1}\rangle^2}{8\pi^2}\frac{2}{t-t_0}\Big(\int_{t_0+\frac{t-t_0}{2}}^t 2\bar{c}_t\frac{2^{\alpha_1/2}}{(t-r)^{1/2-\alpha_1/2}}\|U_{r,t}\nu(.) - U_{r,t}\nu^\epsilon(.)\|_\infty\varrho_2(dr)\Big)^2 \\ &\le\ \frac{c_{t,\nu}}{t-t_0}\Big(\int_\tau^{t_0+\frac{t-t_0}{2}}\frac{1}{(r-\tau)^{1/2-\alpha_1/2}}\|U_{r,t}\nu(.) - U_{r,t}\nu^\epsilon(.)\|_\infty\varrho_2(dr)\Big)^2 \end{aligned}$$

$$
\begin{aligned}
& +\frac{c_{t,\nu}}{t-t_0}\Big(\int_{t_0+\frac{t-t_0}{2}}^{t} \frac{1}{(t-r)^{1/2-\alpha_1/2}}\|U_{r,t}\nu(.)-U_{r,t}\nu^\epsilon(.)\|_\infty \varrho_2(dr)\Big)^2 \\
\le\ & \frac{c_{t,\nu}}{t-t_0}\int_\tau^{t_0+\frac{t-t_0}{2}} \frac{1}{(r-\tau)^{1-\frac{\alpha_1}{2}}}\varrho_2(dr)\int_\tau^{t_0+\frac{t-t_0}{2}} (r-\tau)^{\frac{\alpha_1}{2}}\|U_{r,t}\nu(.)-U_{r,t}\nu^\epsilon(.)\|_\infty^2\varrho_2(dr) \\
& +\frac{c_{t,\nu}}{t-t_0}\int_{t_0+\frac{t-t_0}{2}}^{t} \frac{1}{(t-r)^{1-\frac{\alpha_1}{2}}}\varrho_2(dr)\int_{t_0+\frac{t-t_0}{2}}^{t} (t-r)^{\frac{\alpha_1}{2}}\|U_{r,t}\nu(.)-U_{r,t}\nu^\epsilon(.)\|_\infty^2\varrho_2(dr) \\
\le\ & \frac{c'_{t,\nu}}{t-t_0}\int_\tau^{t_0+\frac{t-t_0}{2}} (r-\tau)^{\alpha_1/2}\|U_{r,t}\nu(.)-U_{r,t}\nu^\epsilon(.)\|_\infty^2\varrho_2(dr) \\
& +\frac{c'_{t,\nu}}{t-t_0}\int_{t_0+\frac{t-t_0}{2}}^{t} (t-r)^{\alpha_1/2}\|U_{r,t}\nu(.)-U_{r,t}\nu^\epsilon(.)\|_\infty^2\varrho_2(dr) \\
\le\ & \frac{c'_{t,\nu}}{t-t_0}\int_\tau^{t_0+\frac{t-t_0}{2}} (t_0+\frac{t-t_0}{2}-\tau)^{\frac{\alpha_1}{2}}\frac{(t-r)^{\frac{\alpha_1}{2}}}{((t-t_0)/2)^{\frac{\alpha_1}{2}}}\|U_{r,t}\nu(.)-U_{r,t}\nu^\epsilon(.)\|_\infty^2\varrho_2(dr) \\
& +\frac{c'_{t,\nu}}{t-t_0}\int_{t_0+\frac{t-t_0}{2}}^{t} (t-r)^{\alpha_1/2}\|U_{r,t}\nu(.)-U_{r,t}\nu^\epsilon(.)\|_\infty^2\varrho_2(dr) \\
\le\ & c_{t,t-t_0,\nu}\int_\tau^t (t-r)^{\alpha_1/2}\|U_{r,t}\nu(.)-U_{r,t}\nu^\epsilon(.)\|_\infty^2\varrho_2(dr)
\end{aligned}
$$

where we used Lemma 4.2$(i)\Rightarrow(ii)$ and Hölder's inequality. Altogether,

$$
\begin{aligned}
& \|U_{\tau,t}\nu(.)-U_{\tau,t}\nu^\epsilon(.)\|_\infty^2 \\
& \quad \le \tilde c_{t,t-t_0,\nu}\Big(\epsilon^\theta+\int_{t_0}^t (t-r)^{\alpha_1/2}\|U_{r,t}\nu(.)-U_{r,t}\nu^\epsilon(.)\|_\infty^2\varrho_2(dr)\Big) \\
& \qquad +\tilde c_{t,t-t_0,\nu}\int_\tau^{t_0}\|U_{r,t}\nu(.)-U_{r,t}\nu^\epsilon(.)\|_\infty^2\varrho_2(dr)
\end{aligned}
$$

holds for all $\tau\in[s,t_0]$. A substitution $\tau=s+t_0-\tau'$ leads to

$$
\begin{aligned}
& \|U_{s+t_0-\tau',t}\nu(.)-U_{s+t_0-\tau',t}\nu^\epsilon(.)\|_\infty^2 \\
& \quad \le \tilde c_{t,t-t_0,\nu}\Big(\epsilon^\theta+\int_{t_0}^t (t-r)^{\alpha_1/2}\|U_{r,t}\nu(.)-U_{r,t}\nu^\epsilon(.)\|_\infty^2\varrho_2(dr)\Big) \\
& \qquad +\tilde c_{t,t-t_0,\nu}\int_s^{\tau'}\|U_{s+t_0-r,t}\nu(.)-U_{s+t_0-r,t}\nu^\epsilon(.)\|_\infty^2\varrho_2(dr)
\end{aligned}
$$

for all $\tau'\in[s,t_0]$. Then, applying Lemma 4.11 to the map

$$
\tau'\mapsto g(\tau'):=\|U_{s+t_0-\tau',t}(.|\nu)-U_{s+t_0-\tau',t}(.|\nu^\epsilon)\|_\infty^2,\qquad s\le\tau'\le t_0
$$

yields in particular (9.66).

Step 2. In order to complete the proof of Lemma 9.33 it is enough to find some $t_0\in(s,t)$ (sufficiently close to t) such that the integral on the r.h.s. of (9.66) tends to 0 as $\epsilon \downarrow 0$. Let us denote the integral term on the r.h.s. of (9.66) by J_3^ϵ. Proceeding as for the estimates of J_1^ϵ and J_2^ϵ we get for arbitrary $\theta \in (0, \alpha_1/2)$:

$$\begin{aligned}
J_3^\epsilon \;\le\; & \int_{t_0}^t (t-r)^{\alpha_1/2} \times \\
& c_t\Big(\frac{1}{(t-r)^{1+\theta}}\epsilon^\theta + \frac{1}{t-r}\int_r^{r+\frac{t-r}{2}} (u-r)^{\alpha_1/2}\|U_{u,t}\nu(.) - U_{u,t}\nu^\epsilon(.)\|_\infty^2 \varrho_2(du) \\
& \qquad + \frac{1}{t-r}\int_{r+\frac{t-r}{2}}^t (t-u)^{\alpha/2}\|U_{u,t}\nu(.) - U_{u,t}\nu^\epsilon(.)\|_\infty^2 \varrho_2(du)\Big)\varrho_2(dr) \\
\le\; & c_t \int_{t_0}^t \frac{1}{(t-r)^{1+\theta-\alpha_1/2}}\varrho_2(dr)\,\epsilon^\theta \\
& + c_t \int_{t_0}^t \frac{(t-r)^{\frac{\alpha_1}{2}}}{t-r}\int_r^{r+\frac{t-r}{2}} (u-r)^{\frac{\alpha_1}{2}}\frac{(t-u)^{\frac{\alpha_1}{2}}}{(t-u)^{\frac{\alpha_1}{2}}}\|U_{u,t}\nu(.) - U_{u,t}\nu^\epsilon(.)\|_\infty^2 \varrho_2(du)\varrho_2(dr) \\
& + c_t \int_{t_0}^t \frac{(t-r)^{\frac{\alpha_1}{2}}}{t-r}\int_{r+\frac{t-r}{2}}^t (t-u)^{\frac{\alpha_1}{2}}\|U_{u,t}\nu(.) - U_{u,t}\nu^\epsilon(.)\|_\infty^2 \varrho_2(du)\varrho_2(dr) \\
\le\; & c_t'\,\epsilon^\theta \\
& + c_t \int_{t_0}^t \frac{(t-r)^{\frac{\alpha_1}{2}}}{t-r}\int_r^{r+\frac{t-r}{2}} \Big(\frac{t-r}{2}\Big)^{\frac{\alpha_1}{2}}\frac{(t-u)^{\frac{\alpha_1}{2}}}{(\frac{t-r}{2})^{\frac{\alpha_1}{2}}}\|U_{u,t}\nu(.) - U_{u,t}\nu^\epsilon(.)\|_\infty^2 \varrho_2(du)\varrho_2(dr) \\
& + c_t \int_{t_0}^t \frac{(t-r)^{\frac{\alpha_1}{2}}}{t-r}\int_{r+\frac{t-r}{2}}^t (t-u)^{\frac{\alpha_1}{2}}\|U_{u,t}\nu(.) - U_{u,t}\nu^\epsilon(.)\|_\infty^2 \varrho_2(du)\varrho_2(dr) \\
\le\; & c_t'\,\epsilon^\theta + c_t\int_{t_0}^t \frac{1}{(t-r)^{1-\alpha_1/2}}\int_r^t (t-u)^{\alpha_1/2}\|U_{u,t}\nu(.) - U_{u,t}\nu^\epsilon(.)\|_\infty^2 \varrho_2(du)\varrho_2(dr) \\
\le\; & c_t'\,\epsilon^\theta + c_t\int_{t_0}^t \frac{1}{(t-r)^{1-\alpha_1/2}}\int_{t_0}^t (t-u)^{\alpha_1/2}\|U_{u,t}\nu(.) - U_{u,t}\nu^\epsilon(.)\|_\infty^2 \varrho_2(du)\varrho_2(dr) \\
=\; & c_t'\,\epsilon^\theta + c_t\int_{t_0}^t \frac{1}{(t-r)^{1-\alpha_1/2}}\,J_3^\epsilon\,\varrho_2(dr) \;\le\; c_t'\,\epsilon^\theta + c_t''(t-t_0)^{\alpha_1/2+\alpha_2-1}J_3^\epsilon.
\end{aligned}$$

where we used Lemma 4.4(i) in the last line. For t_0 sufficiently close to t (i.e. for t_0 satisfying $t - (2c_t'')^{-(\alpha_1/2+\alpha_2-1)} < t_0 < t$) we obtain $J_3^\epsilon \le c_t'\,\epsilon^\theta + \frac{1}{2}J_3^\epsilon$ whence J_3^ϵ converges to 0 as $\epsilon \downarrow 0$. This completes the proof of Lemma 9.33. □

We are now in the position to prove (9.63). By the weak convergence of ν^ϵ to ν ($\epsilon \downarrow 0$), Theorem 9.27 and dominated convergence as well as (9.9) and Lemma 9.33 we obtain

$$\begin{aligned}
\mathbb{E}_{s,\eta}\Big[e^{-\langle\nu,X_t\rangle}\Big] &= \mathbb{E}_{s,\eta}\Big[\lim_{\epsilon\downarrow 0} e^{-\langle\nu^\epsilon,X_t\rangle}\Big] = \lim_{\epsilon\downarrow 0}\mathbb{E}_{s,\eta}\Big[e^{-\langle\nu^\epsilon,X_t\rangle}\Big] = \lim_{\epsilon\downarrow 0}\mathbb{E}_{s,\eta}\Big[e^{-\langle\bar{X}_t,P_\epsilon\nu\rangle}\Big] \\
&= \lim_{\epsilon\downarrow 0} e^{-\langle\eta,U_{s,t}P_\epsilon\nu(.)\rangle} = \lim_{\epsilon\downarrow 0} e^{-\langle\eta,U_{s,t}\nu^\epsilon(.)\rangle} = e^{-\langle\eta,U_{s,t}\nu(.)\rangle}.
\end{aligned}$$

This completes the proof of Theorem 9.31.

9.11 First and second moments of the density field

From Theorem 9.27 we know that the 1-dimensional weakly continuous catalytic SBM $\bar{X}$ possesses a jointly continuous density field X whenever the catalyst $\varrho(dtdx)$ satisfies condition (A). We here present a formula for the first and the second moments of the random variable $X_t(x)$. Recall that we defined $P_t\eta(x) := \int_{\mathbb{R}} p_t(x,y)\eta(dy)$ for every $t > 0$, $x \in \mathbb{R}$ and $\eta \in \mathcal{M}_f(\mathbb{R})$.

Theorem 9.34 [MOMENTS] *Let $\varrho(dtdx)$ satisfy condition* (A) *and $\bar{X} = [\bar{X}, \mathbb{P}_{s,\eta} : s \geq 0, \eta \in \mathcal{M}_f(\mathbb{R})]$ denote the corresponding canonical continuous catalytic SBM. Pick $s \geq 0$ and $\eta \in \mathcal{M}_f(\mathbb{R})$. The jointly continuous density field $(X_t : t > s)$ of $(\bar{X}_t : t > s)$, which exists under $\mathbb{P}_{s,\eta}$ by Theorem 9.27, satisfies for all $t > s$ and $x \in \mathbb{R}$:*

$$\begin{aligned}
\mathbb{E}_{s,\eta}\left[X_t(x)\right] &= P_{t-s}\eta(x), \\
\mathbb{E}_{s,\eta}\left[X_t^2(x)\right] &= (P_{t-s}\eta)^2(x) + \int_{\mathbb{R}}\int_s^t\int_{\mathbb{R}} p_{r-s}(y,z)p_{t-r}^2(x,y)\varrho(drdy)\eta(dz).
\end{aligned}$$

We need the following lemma for the proof of Theorem 9.34.

Lemma 9.35 *For $t > s$ and $x \in \mathbb{R}$ we have:*

(i) $\lim_{\epsilon\downarrow 0} \|p_{t-s}(x,.) - p_{t-s+\epsilon}(x,.)\|_\infty = 0,$

(ii) $\lim_{\epsilon\downarrow 0} \| \int_s^t \int_{\mathbb{R}} p_{r-s}(y,.)p_{t-r}^2(x,y)\varrho(drdy) - \int_s^t \int_{\mathbb{R}} p_{r-s}(y,.)p_{t-r+\epsilon}^2(x,y)\varrho(drdy)\|_\infty = 0.$

Proof Using Lemma 4.5(i) we obtain for all $y \in \mathbb{R}$:

$$|p_{t-s}(x,y) - p_{t-s+\epsilon}(x,y)| \leq c\int_{t-s}^{t-s+\epsilon} \frac{1}{u}p_{2u}(x,y)du \leq c\frac{1}{t-s}\,\epsilon^{1/2}.$$

This proves part (i). By means of Lemma 4.5(i), Lemma 4.2 (i) $\Rightarrow$ (ii) and Lemma 4.4(i) we also obtain for all $z \in \mathbb{R}$ and some $\theta \in (0, \alpha_1/2 + \alpha_2 - 1)$:

$$\begin{aligned}
&\Big|\int_s^t\int_{\mathbb{R}} p_{r-s}(y,z)p_{t-r}^2(x,y)\varrho(drdy) - \int_s^t\int_{\mathbb{R}} p_{r-s}(y,z)p_{t-r+\epsilon}^2(x,y)\varrho(drdy)\Big| \\
&\leq \int_s^t\int_{\mathbb{R}} p_{r-s}(y,z)|p_{t-r}^2(x,y) - p_{t-r+\epsilon}^2(x,y)|\varrho(drdy) \\
&\leq \int_s^t\int_{\mathbb{R}} p_{r-s}(y,z)(p_{t-r}(x,y) + p_{t-r+\epsilon}(x,y))|p_{t-r}(x,y) - p_{t-r+\epsilon}(x,y)|\varrho(drdy) \\
&\leq \int_s^t\int_{\mathbb{R}} p_{r-s}(y,z)\frac{2}{\sqrt{2\pi(t-r)}}\int_{t-r}^{t-r+\epsilon}\frac{1}{u}p_{2u}(x,y)du\varrho(drdy) \\
&\leq \int_s^t \frac{1}{\sqrt{2\pi(r-s)}}\frac{2}{\sqrt{2\pi(t-r)}}\int_{t-r}^{t-r+\epsilon}\frac{1}{u}\frac{1}{\sqrt{4\pi u}}\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4u}}\varrho_1(r,dy)dudr \\
&\leq \frac{1}{\pi}\int_s^t \frac{1}{\sqrt{r-s}}\frac{1}{\sqrt{t-r}}\int_{t-r}^{t-r+\epsilon}\frac{1}{u}\frac{1}{\sqrt{4\pi u}}\,\bar{c}_t(4u)^{\alpha_1/2}du\varrho_2(dr) \\
&\leq c_t'\int_s^t \frac{1}{\sqrt{r-s}}\frac{1}{(t-r)^{1-\alpha_1/2+\theta}}\int_{t-r}^{t-r+\epsilon}\frac{1}{u^{1-\theta}}du\varrho_2(dr)
\end{aligned}$$

$$
\begin{aligned}
&\le\; c_t' \int_s^t \frac{1}{\sqrt{r-s}} \frac{1}{(t-r)^{1-\alpha_1/2+\theta}}\; c'\epsilon^\theta \varrho_2(dr) \\
&\le\; c_t'' \Big(\frac{1}{(\frac{t-s}{2})^{1-\theta}} \int_s^{s+\frac{t-s}{2}} \frac{1}{\sqrt{r-s}} \varrho_2(dr) + \frac{1}{\sqrt{\frac{t-s}{2}}} \int_{s+\frac{t-s}{2}}^t \frac{1}{(t-r)^{\alpha_1/2-1-\theta}} \varrho_2(dr) \Big)\, \epsilon^\theta \\
&\le\; c_t''' \Big(\frac{1}{(\frac{t-s}{2})^{1-\theta}} (t-s)^{\alpha_2-1/2} + \frac{1}{\sqrt{\frac{t-s}{2}}} (t-s)^{\alpha_1/2+\alpha_2-1-\theta} \Big)\, \epsilon^\theta \;\le\; c_{t-s,t}\, \epsilon^\theta
\end{aligned}
$$

which implies assertion (ii) (since $\alpha_2 > 1/2$ whenever condition (A) is fulfilled). □

Proof (of Theorem 9.34) As in Steps 1 and 2 of the proof of Theorem 9.27 one can show that $\langle \bar{X}_t, p_\epsilon(x,.)\rangle$ converges in $L^2(\mathbb{P}_{s,\eta})$ to $X_t(x)$ (as $\epsilon \downarrow 0$) for every $t > s$ and $x \in \mathbb{R}$. In particular,

(a) $\mathbb{E}_{s,\eta}[X_t(x)] = \lim_{\epsilon\downarrow 0} \mathbb{E}_{s,\eta}[\langle \bar{X}_t, p_\epsilon(x,.)\rangle]$

(b) $\mathbb{E}_{s,\eta}[X_t^2(x)] = \lim_{\epsilon\downarrow 0} \mathbb{E}_{s,\eta}[\langle \bar{X}_t, p_\epsilon(x,.)\rangle^2]$.

The first moment formula in Theorem 9.34 follows from (9.23), (a) and Lemma 9.35(i). The second moment formula in Theorem 9.34 follows from (9.23), (b), Lemma 9.35(ii) as well as Lemma 9.35(i) and

$$
\begin{aligned}
&|\langle \eta, p_{t-s}(x,.)\rangle^2 - \langle \eta, p_{t-s+\epsilon}(x,.)\rangle^2| \\
&\quad\le\; (\langle \eta, p_{t-s}(x,.)\rangle + \langle \eta, p_{t-s+\epsilon}(x,.)\rangle)\langle \eta, \mathbf{1}\rangle \|p_{t-s}(x,.) - p_{t-s+\epsilon}(x,.)\|_\infty \\
&\quad\le\; c_{t-s,\eta}\, \|p_{t-s}(x,.) - p_{t-s+\epsilon}(x,.)\|_\infty.
\end{aligned}
$$

□

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Index

Notation

$a \vee b$, $a \wedge b$	maximum, respectively minimum, of a and b,
$\mathcal{A}(S)$	algebra of relatively compact sets from $\mathcal{B}(S)$, p.13
$\|A\|$	diameter of set A, p.8
$B(I,E)$	E-valued Borel measurable functions on I,
$B(I)$	$\mathbb{R}$-valued Borel measurable functions on I,
$B_b(I)$	bounded functions in $B(I)$,
$\mathcal{B}(S)$	Borel σ-algebra on S, p.7
$B(x,r)$, $B[x,r]$	open, respectively closed, ball around x with radius r, p.10, 12
CAF	continuous additive functional of a Markov process, p.41
$c, \bar{c}, \tilde{c}, c', c''c'''$	positive finite constants that may vary from place to place; possible subscripts stress the dependence on these subscripts,
$C(I,E)$	E-valued continuous functions on I,
$C(I)$	$\mathbb{R}$-valued continuous functions on I,
$C_0(I)$	functions in $C(I)$ that vanish "at infinity", p.40
$C_+(I)$, $C^+(I)$	non-negative functions in $C(I)$,
$C_b(I)$	bounded functions in $C(I)$,
$C_b^n(\mathbb{R}^d)$	functions in $C_b(\mathbb{R}^d)$ with bounded and continuous derivative up to order n,
$C_b^{1,2}([0,\infty)\times\mathbb{R}^d)$	functions in $C_b([0,\infty)\times\mathbb{R}^d)$ that are differentiable once in time and twice in space (with bdd. and contin. derivatives), p.155
$C_{b,\infty}^{1,2}([0,\infty)\times\mathbb{R}^d)$	functions in $C_b^{1,2}([0,\infty)\times\mathbb{R}^d)$ for which $t \mapsto f(t,.)$ is continuous w.r.t. $\\|.\\|_\infty$, p.155
$C_b^\infty(\mathbb{R}^d)$	functions in $C_b(\mathbb{R}^d)$ with bounded and continuous derivatives of any order, p.159
$C_b^\infty([0,t]\times\mathbb{R}^d)$	functions in $C_b([0,t]\times\mathbb{R}^d)$ with bounded and continuous derivatives of any order, p.159
$C_c(\mathbb{R}^d)$	functions in $C(\mathbb{R}^d)$ with compact support,
$C_c^\infty(\mathbb{R}^d)$	functions in $C_c(\mathbb{R}^d)$ with derivatives of any order,
$C_{int}^+(\mathbb{R}^d)$	Lebesgue-integrable functions in $C^+(\mathbb{R}^d)$, p.172
$C_{Lip}(\hat{\mathbb{R}}^d)$	Lipschitz continuous functions on $\hat{\mathbb{R}}^d$, p.150
$C_{rap}(\mathbb{R}^d)$	rapidly decreasing functions in $C(\mathbb{R}^d)$, p.22
$C_{tem}(\mathbb{R}^d)$	tempered functions in $C(\mathbb{R}^d)$, p.22
$C_{rap}^{1,2}([0,\infty)\times\mathbb{R}^d)$	$C_{rap}(\mathbb{R}^d)$-valued continuous functions with ..., p.88, 114
$C_{[B,\mu]}$, CCP	continuous collision process of B and μ, p.126

$C_{[\bar{X},\varrho]}$	collision measure of $\bar{X}$ and ϱ, p.152
$\mathcal{C}_\lambda$, $\mathcal{C}_\lambda^n$	λ-Cantor measure on $\mathbb{R}$, respectively $\mathbb{R}^n$, p.11
$C(\lambda)$	λ-Cantor set, p.11
Δ, Δ_ϵ	Laplacian, approximate Laplacian, p.70, 95, 115
$D(I,E)$	E-valued cadlag functions on I,
$d_{\mathcal{M}(S)}$, $d_{\mathcal{M}(S)}$	metric on $\mathcal{M}(S)$, respectively $\mathcal{M}_f(S)$, p.12, 14
d_{rap}, d_{tem}	metric on $C_{rap}(\mathbb{R}^d)$, respectively $C_{rap}(\mathbb{R}^d)$, p.23
$d_{tem,\infty}$	metric on $C([0,\infty), C_{tem}(\mathbb{R}^d))$, p.26
$\dim A$	Hausdorff dimension of A, p.9
d_∞	metric on $C(I,E)$, p.22
E^I	functions from I to E, p.19
$\mathcal{E}^I$	σ-algebra on E^I generated by $\bar{\pi}_t, t \in I$, p.19
$\bar{\mathcal{E}}^I$	$\bar{\mathcal{E}}^I = \mathcal{E}^I \cap C(I,E)$, p.21
$\mathbb{F}$	any subspace of $C(\mathbb{R}^d)$, p.77
$\mathcal{F}_{pred}$, $\bar{\mathcal{F}}_{pred}$	σ-algebra generated by simple processes, p.33, 74
$(\mathcal{F}_t)$	filtration, p.20
$(\tilde{\mathcal{F}}_t^{\mathbb{P}})$	$\mathbb{P}$-completion of $(\mathcal{F}_t)$, p.20
$(\bar{\mathcal{F}}_t)$, $(\bar{\mathcal{F}}_t^{\mathbb{P}})$	usual augmentation of $(\mathcal{F}_t)$, p.20
$(\mathcal{F}_{[s,t]}^X)$, $(\mathcal{F}_t^X)$	natural filtrations induced by the process X, p.20
$\langle \phi, \psi \rangle$	integral $\int \phi(x)\psi(x)dx$,
$\Phi, \Phi_2, \Phi_3, \Phi'$	functionals on $\mathcal{P}$, respectively $\mathcal{P}'$, p.91, 114
$f \cdot M$	Walsh-integral of f w.r.t. M, p.75
$\mathcal{H}^\alpha$	α-dimensional Hausdorff measure, p.9
$h_{s,t}(x)$	characteristic of CAF, p.42
$\mathbb{I}$	identity, $\mathbb{I}(f) := f$,
$I^M(X)$	Itô-integral of X w.r.t. M, p.69
$L_{\mathbb{P}}$	Laplace transform of probability measure $\mathbb{P}$ on $\mathcal{M}_f(S)$, p.36
$\mathcal{L}^n$	Lebesgue measure on $\mathbb{R}^n$, p.8
$\mathcal{L}^2(M)$	progressively measurable processes X with $[X]^M < \infty$, p.68
$[.]^M$, $[.]_t^M$, $[.]$, $[.]_t$	metric on $\mathcal{L}^2(M)$, p.68
$\langle M \rangle$	quadratic variation process of M, p.33
$\langle M, M' \rangle$	covariation process of M and M', p.34
$\langle M \rangle(dtdx)$	quadratic variation measure of M, p.73

$\mathcal{M}$, $\mathcal{M}_c$, $\mathcal{M}^2$	martinales, continuous, square-integrable, p.32
$\|.\|$, $\|.\|_t$	metric on $\mathcal{M}^2$, p.32
$\mathcal{M}_{loc}$, $\mathcal{M}_{c,loc}$	local martingales, continuous, p.34
$\mathcal{M}(S)$	Radon measures on S, p.12
$\mathcal{M}_f(S)$	finite measures on S, p.13
$\mathcal{M}_1(S)$	probability measures on S, p.13
$\mathcal{M}_{uni}(\mathbb{R}^d)$	uniformly bounded measures on $\mathbb{R}^d$, p.16
$\mathfrak{M}(S)$, $\mathfrak{M}_f(S)$	σ-algebra on $\mathcal{M}(S)$, respectively $\mathcal{M}_f(S)$, p.15
$M_{s,.}(\psi)$	process wanted to be a martingale, p.155
$\langle \mu, \psi \rangle$	integral $\int \psi(x)\mu(dx)$,
$\mu_{s,t}(x,.)$	transition probability, p.38
$\mathcal{N}_\mu$	μ-negligible sets, p.8
N_μ	μ-null sets, p.8
$N(0,v)$	normal distribution with mean zero and variation v,
$\|.\|_\infty$	supremum norm, $\|f\|_\infty := \sup_t f(t)$,
$\mathcal{P}$	predictable processes u with $\|u\|_{\lambda,T,m} < \infty$ for all $\lambda, T > 0$ and $m \geq 1$, p.90
$\|.\|_{\lambda,T,m}$	seminorm on $\mathcal{P}$, p.90
$\mathbb{P}_{s,x}$	law of Markov process starting from x at time s, p.38
$\mathcal{P}_s$	predictable processes X with $\|\|\|X\|\|\|_{s,T} < \infty$ $\forall T > s$, p.165
$\|\|\|.\|\|\|_{s,T}$	seminorm on $\mathcal{P}_s$, p.165
$\mathcal{P}^2(M)$	predictable processes X with $\|[X]\|^M < \infty$, p.74
$\|[.]\|^M$, $\|[.]\|_t^M$, $\|[.]\|_t$	metric on $\mathcal{P}^2(M)$, p.74
(P_t), (P_t^ϵ)	heat semigroup, approximate heat semigroup, p.55, 95, 115
$p_t(x,y)$	heat kernel, p.49
$\mathbb{P}_X$	law of process X, p.19
π_t, $\bar{\pi}_t$	projection $\pi_t(f) := f(t)$, its restriction to $C(I,E)$, p.19, 21
$\bar{\pi}_A$, $\bar{\pi}_\psi$	projections $\bar{\pi}_A(\mu) := \mu(A)$, $\bar{\pi}_\psi(\mu) := \langle \mu, \psi \rangle$, p.12, 13
(Q_t^ϵ), $q_t^\epsilon(x,y)$	semigroup corresponding to (P_t^ϵ), its kernel, p.95, 115
$\hat{\mathbb{R}}^d$, $[0,\infty]$	Alexandrov's one-point compactification of $\mathbb{R}^d$, resp. $[0,\infty)$, p.15, 149
SBM	super-Brownian motion,
$\sigma(\mathcal{G})$	σ-algebra generated by $\mathcal{G}$, p.7
supp(μ)	closed support of measure μ, p.10

$\mathfrak{S}$, $\bar{\mathfrak{S}}$	simple processes, p.69, 74
θ_s	shift operator on the path space, $\theta_s f(.) := f(s + .)$,
$(U_{s,t})$	log-Laplace semigroup associated with $\bar{X}$, p.135
$w^{\varrho}_{.}$	square-integrable martingale with $\langle w^{\varrho} \rangle_t = \varrho([0,t])$, p.66
$w^{\varrho}(.,.)$	two-parameter version of $w^{\varrho}_{.}$, p.71
$\bar{W}^{\varrho}$	white noise measure with intensity measure ϱ, p.65
W^{ϱ}, $W^{\varrho}(dtdx)$	orthogonal martingale measure with $\langle W^{\varrho} \rangle = \varrho$, p.78
$\bar{X}$	catalytic super-Brownian motion, p.135